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Weil's Representation and the Spectrum of the Metaplectic Group

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For Mary

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Introduction

These notes are expanded from my earlier paper [8] mimeographed at Cornell in July 1974. In addition to including corrections and revisions for [8], the present notes contain new results and insights obtained during the past year and a half. Some of these results have already been described briefly in [9] and [10]. Others are due to Roger Howe and Pierre Cartier. In particular, parts of Subsections 2.4, 5.5, and 6.2 are taken from correspondence with Howe, and parts of Subsections 2.5 and 3.1 were suggested by Cartier after his critical reading of the original manuscript. My indebtedness to both goes beyond acknowledgement of their suggestions within the text.

The goal of these Notes is a general theory of automorphic form for the metaplectic group. I am indebted to Robert Langlands for inspiring this project and giving freely of his ideas. I am also grateful to Paul Sally for his collaboration on [10] (cf. Subsections 4.4 and 5.5), and my colleagues at Cornell, especially Kenneth Brown, Robert Strichartz, and William Waterhouse, for many helpful conversations.

The theory of automorphic forms on the metaplectic group described in these Notes is still in its infancy. Moreover, many of the results, if not incomplete, are in preliminary form. I am grateful to all the people named above for helping me realize this.

The expeditious typing of these Notes was done by Joanne Lewis, Arletta Havlik and Esther Monroe.

S. Gelbart

November 1975

§1. Background and Summary of Results.

The metaplectic group was first introduced by Weil in [47]. His purpose was to reformulate Siegel's analytic theory of quadratic forms in group theoretic terms. The motivation for our investigation comes from more recent works of Shimura and Kubota. Our purpose is to describe the spectrum of the metaplectic group modulo its subgroup of rational points and to relate this spectrum to the theory of automorphic forms for GL(2).

Our work relies heavily upon Weil's. However, as already suggested, it is more closely related to important recent discoveries of Shimura and Kubota which we shall now briefly describe.

Fix k to be an odd positive integer, N a positive integer divisible by 4, and χ a character of the integers defined modulo N. Put

$$\Gamma_0(N) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(N)\},$$

and

$$\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i\, n^2 z).$$

Then

$$|\theta(\gamma z)/\theta(z)| = |cz+d|^{1/2}$$

for all $\gamma \in \Gamma_0(N)$. (Hecke [16], pp. 919-940.)

In [38] Shimura deals with cusp forms $f(z)$ satisfying the identity

$$f(\gamma z) = \chi(d)[\theta(\gamma z)/\theta(z)]^k f(z)$$

for all $\gamma \in \Gamma_0(N)$. Such functions comprise the space of <u>classical cusp forms of weight $k/2$, character χ, and "θ-multiplier system" $[\theta(\gamma z)/\theta(z)][cz+d]^{-1/2}$</u>. We denote this space by $S_{k/2}(N,\chi)$. Functions in it arise naturally in number theory from the study of partitions and quadratic forms in an odd number of variables.

To study functions in $S_{k/2}(N,\chi)$ one introduces certain linear operators $T^N_{k,\chi}(m)$ (following Hecke in the case of forms of integral weight). These $T(m)$ operate in $S_{k/2}(N,\chi)$ for all integers m. However, $T^N_{k,\chi}(m)$ operates as the zero operator if m is not a square and $(m,N) = 1$. This curious fact seems to have discouraged Hecke from beginning a systematic study of such forms along the lines of his already successful theory for forms of integral weight. Nonetheless, Shimura established the following provocative result.

Suppose $f(z) = \sum a(n)\exp(2\pi i\, n\, z)$ in $S_{k/2}(N,\chi)$ is an eigenfunction for every $T^N_{k,\chi}(p^2)$, say

$$T(p^2)f = \lambda(p)f.$$

Suppose also that $k \geq 3$. Then

$$\Big(\sum_{n=1}^{\infty} a(n^2)n^{-s}\Big)L^*\Big(\chi, 1 - \frac{k-1}{2} - s\Big)$$

$$= \prod_p [1-\lambda(p)p^{-s} + \chi(p)^2 p^{k-2-2s}]^{-1},$$

where $L^*(\chi,\cdot)$ is essentially the Dirichlet L-series associated to χ; furthermore, and more significantly, the inverse Mellin transform of

$$\sum_1^{\infty} A(n)n^{-s} = \prod_p [1-\lambda(p)p^{-s} + \chi(p)^2 p^{k-2-2s}]^{-1},$$

namely

$$F(z) = \sum_{n=1}^{\infty} A(n)\exp(2\pi i\, n\, z),$$

satisfies

$$F(\gamma z) = \chi(d)^2(cz+d)^{k-1}F(z),$$

for all $\gamma \in \Gamma_0(N_0)$. (Here N_0 depends only on N and χ and usually equals $N/2$.)

The significance of Shimura's result is that it establishes a correspondence between forms in $S_{k/2}(N,\chi)$ and forms of integral (actually even) weight $k-1$. Note that $T(p)F = \lambda(p)F$. Thus this correspondence preserves eigenvalues for the Hecke ring.

The success of Shimura's theory leads one to ask several important questions. For example, can Shimura' correspondence be defined without recourse to L-functions? In particular, what is its group theoretic interpretation? Is the correspondence one-to-one?

In the first part of this paper I shall interpret forms of half-integral weight as irreducible representations of the metaplectic group "defined over $\mathbb{Q}$." Thus, the full weight of the representation theory of this group can be used to discuss these questions.

Kubota's results also concern forms of "half-integral weight". To describe them, fix an imaginary quadratic field

$$F = \mathbb{Q}(\sqrt{-d}),$$

denote by O the ring of integers of F, and by $(\frac{a}{b})$ the quadratic power residue symbol in F. Let $\Gamma(N)$ denote the congruence subgroup mod N of $SL(2,O)$ and define the function $\chi(\gamma)$ on $\Gamma(N)$ by

$$\chi([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = \begin{cases} (\frac{c}{d}) & \text{if } c \neq 0 \\ 1 & \text{otherwise .} \end{cases}$$

The starting point for Kubota's investigation is his discovery that χ is a character of $\Gamma(N)$ (provided N, as before, is congruent to 0 mod 4). This result is equivalent to the reciprocity law for $(\frac{a}{b})$ modulo the fact that $\mathbb{Q}\sqrt{-d}$ is totally imaginary. (See [19] for the original proof and Subsection 2.2 of this paper for the case of arbitrary number fields.

Now let H denote the three-dimensional quaternionic upper half-space whose points are of the form $u = (z,v) = [\begin{smallmatrix} z & -v \\ v & \bar{z} \end{smallmatrix}]$, $(z \in \mathbb{C},\ v > 0)$. The operation of $\sigma = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in SL(2,\mathbb{C})$ on H is

given by

$$\sigma_u = (au+b)(cu+d)^{-1}$$

(identifying $t \in \mathbb{C}$ with the matrix $\left[\begin{smallmatrix} t & 0 \\ 0 & \frac{}{t} \end{smallmatrix}\right]$) so H is isomorphic to $SL(2,\mathbb{C})/SU(2)$. The quotient space

$$\Gamma(N)/H$$

is of finite volume.

In [21] Kubota considers modular functions on H with respect to $\Gamma(N)$ and character χ; his special interest is in

$$E(u,s) = \sum_{\Gamma_\infty \backslash \Gamma(N)} \chi(\gamma) v(\gamma u)^{s+1}$$

where s is a complex variable, $v(u) = v$ if $u = (z,v)$, and Γ_∞ is the upper triangular subgroup of $\Gamma(N)$.

The series defining $E(u,s)$ is an Eisenstein series. Although it converges only for $Re(s) > 1$, $E(u,s)$ itself has an analytic continuation to the whole s-plane and a pole of the first order at $s = 1/2$. The residue $\theta(u)$ of $E(u,s)$ at $s = 1/2$ satisfies

$$\theta(\gamma u) = \chi(\gamma)\theta(u)$$

for all $\gamma \in \Gamma(N)$ and defines a square-integrable automorphic form which is not a cusp form. In fact θ defines a theta-function. This function can not be lifted to the adele group of $GL(2)$ over $\mathbb{Q}(\sqrt{-d})$. Rather it defines a function on a certain two-fold covering of this group, the metaplectic group of §2.

One of Kubota's principal results is a computation of the eigenvalues of $\theta(u)$ with respect to Hecke's ring. This result generalizes the fact that the eigenvalues of the classically defined holomorphic Eisenstein series in $\{Im(z) > 0\}$ are given by arithmetic functions such as the sum of the divisors of an integer.

As already suggested, one purpose of our investigation is

to reformulate and extend the results of Shimura and Kubota in the context of a general theory of automorphic forms for the metaplectic group. In the context of this general theory, the results of Shimura and Kubota appear to be closely related. Thus it is hoped we have shed light on both of them. In particular, Kubota's $\theta(u)$ becomes a tensor product of certain "extraordinary" representations of the local groups $\overline{G_v}$ defined at each place of F. The corresponding computation of eigenvalues then becomes part of the spherical function theory of these groups.

In addition to working over arbitrary number fields, our methods are new. In that our point of view is representation-theoretic we follow Jacquet-Langlands ([18]) rather closely. In fact, a second goal of our program is to develop a theory for the metaplectic group analogous to Jacquet-Langlands' treatment of Hecke theory for $GL(2)$. The possibility of such a theory was already suggested by Shimura in [38].

To be more precise, let F denote an arbitrary global field and $\mathbb{A}$ its ring of adeles. Let G denote GL_2 regarded as an algebraic group over F. The <u>metaplectic group</u> $\overline{G_{\mathbb{A}}}$ is a central extension of $GL_2(\mathbb{A})$ by Z_2, the group of square roots of unity. Thus

$$1 \to Z_2 \to \overline{G_{\mathbb{A}}} \to GL_2(\mathbb{A}) \to 1$$

is exact and Z_2 is a trivial $GL_2(\mathbb{A})$-module. The crucial property of this extension is that it splits over the subgroup of rational points $GL_2(F)$ and <u>the fundamental problem is to decompose the natural unitary representation of</u> $\overline{G_{\mathbb{A}}}$ <u>in</u> $L^2(G_F \backslash \overline{G_{\mathbb{A}}})$.

Suppose we denote this representation by $\overline{\overline{T}}$. Then

$$\overline{\overline{T}}(g_0)\varphi(g) = \varphi(g\, g_0)$$

for all g, $g_0 \in \overline{G_{\mathbb{A}}}$, and $\varphi \in L^2(G_F \backslash \overline{G_{\mathbb{A}}})$. Note that $L^2(G_F \backslash \overline{G_{\mathbb{A}}})$,

as a $\overline{G_{\mathbb{A}}}$-module, is the direct sum of $L^2(G_F\backslash\overline{G}_{\mathbb{A}}/Z_2)$ (which is just $L^2(G_F\backslash GL_2(\mathbb{A}))$) and the space of functions $\overline{L}^2(G_F\backslash\overline{G_{\mathbb{A}}})$ satisfying

$$\varphi(\zeta g) = \zeta\omega(g), \qquad \zeta \in Z_2)$$

("genuine" functions on $\overline{G_{\mathbb{A}}}$). Thus

$$\overline{\overline{T}} = T \oplus \overline{T},$$

where constituents of T correspond to automorphic forms on $GL(2)$.

The constituents of T comprise the subject matter of Jacquet-Langlands' treatment of Hecke's theory of forms of integral weight. The irreducible constituents of $\overline{T}$, on the other hand, are what we call "genuine automorphic representations of the metaplectic group", or "generalized automorphic forms of half-integral weight over F". This terminology is apt since the forms considered by Shimura correspond to special forms on $\overline{G_{\mathbb{A}}}$ defined over $\mathbb{Q}$. (This observation is the starting point for our theory; see Subsection 3.1 for details.) The Eisenstein series considered by Kubota lead to forms which are defined over a totally imaginary field F, and which occur outside the space of cusp forms.

The main emphasis of our theory is on relations between automorphic forms on the metaplectic group and automorphic forms on $GL(2)$, i.e. between constituents of $\overline{T}$ and constituents of T. Eventually we shall define a map S between arbitrary (i.e. not necessarily automorphic) representations of $\overline{G_{\mathbb{A}}}$ non-trivial on Z_2 and representations of $G_{\mathbb{A}}$. This map will be consistent with Shimura's when restricted to automorphic forms on $\overline{G_{\mathbb{A}}}$. Moreover, it will be one-to-one, and its definition will be entirely representation theoretic.

A correspondence D^3 between certain representations of $G_{\mathbb{A}}$ and $\overline{G_{\mathbb{A}}}$ is constructed completely independently of S using Weil's theory of the metaplectic group. Motivated by recent results of S. Niwa [30] and T. Shintani [41] we collect evidence for the

assertion that

$$D^3(S(\overline{\pi})) = \overline{\pi}$$

whenever $S(\overline{\pi})$ is in the domain of D^3. The validity of this hypothesis is one of the focal points of our theory. We also describe other features of our theory and explain its connection with the results of Shimura and Kubota. All our results lend support to the assertion that forms on $GL(2)$ and the metaplectic group are inextricably connected.

A more precise summary of the contents of this paper now follows.

In Section 2 we analyze the metaplectic coverings of $GL(2)$ over local and global fields. In constructing these groups, we first follow Kubota, then Weil. Whereas Kubota's construction involves an explicit factor set and is well-suited for basic computations, Weil's construction is entirely representation-theoretic and hence particularly crucial to our approach. In Subsections 2.4 and 2.5 we explain how Weil's construction leads to a general philosophy which not only yields the map D alluded to above but also a correspondence D^1 between automorphic forms on $\overline{G_{\mathbb{A}}}$ and automorphic forms on GL_1. These matters are dealt with in detail in Sections 4,5, and 6. Roughly speaking, to each Weil representation r of $\overline{G_{\mathbb{A}}}$ there corresponds a quadratic form q and to each q there is attached a correspondence D^q between irreducible constituents of r_q and constituents of the natural representation of the orthogonal group of q in the space of r_q. In particular,

$$q_3(x_1,x_2,x_3) = x_1^2 - x_2^2 - x_3^2$$

yields the map D^3 alluded to above, while

$$q_1(x) = x^2$$

relates to Kubota's results and D^1.

In Section 3 we introduce the subject matter proper of this paper. Some general features of the decomposition of $L^2(G_F\backslash\overline{G}_{\mathbb{A}})$ are sketched and the connections between constituents of this decomposition and the functions considered by Shimura and Kubota are described in detail. Some miscellaneous results which will be used later on are also collected here. Our basic observation in Section 3 is that one can make sense out of the relation

$$\overline{\pi} = \otimes\, \overline{\pi}_v$$

for certain irreducible "genuine" representations of $\overline{G}_{\mathbb{A}}$. (Note that $\overline{G}_{\mathbb{A}}$ will <u>not</u> be a restricted direct product of the local covering groups $\overline{G}_v$.) This result is important since it reduces global questions to local ones. In particular, in describing the map S alluded to above, one is lead to the study of certain local maps

$$S_v\colon\ \overline{\pi}_v \to \pi_v$$

for each place v of F. In this Section we also isolate the non-cuspidal spectrum of $L^2(G_F\backslash\overline{G}_{\mathbb{A}})$ using the theory of Eisenstein series on $\overline{G}_{\mathbb{A}}$.

In Section 4 we treat the local map for the archimedean places in complete detail. To certain pairs of quasi-characters (μ_1,μ_2) of $F_v = \mathbb{R}$ or $\mathbb{C}$, irreducible representations of $\overline{G}_v$ as well as G_v are described. If $\overline{\pi}_v$ is attached to (μ_1,μ_2) then the map S_v is defined by setting $S_v(\overline{\pi}_v)$ equal to the irreducible representation of G_v attached to (μ_1^2,μ_2^2). This is consistent with Shimura's map since a discrete series representation of $\overline{G}_v$ with lowest weight $\frac{k}{2}$ is mapped to a discrete series representation of G_v with lowest weight $k-1$. (It is consistent with Kubota since it attaches the trivial representation of $G_{\mathbb{C}}$ to the complementary series representation of index $s = 1/2$.)

In Subsection 4.4 we describe how the Weil representation r_1 attached to the quadratic form q_1 decomposes over $\mathbb{R}$. In general, the problem of decomposing Weil representations attached to quadratic forms in an odd number of variables has not yet received the attention it deserves. (See, however, [52].) The most significant result in Section 4 concerns the construction of a map D_v^3 inverse to S_v. This involves the decomposition of the Weil representation r_3 attached to the quadratic form $q_3(x_1,x_2,x_3)$. The orthogonal group of this form is essentially $G_{\mathbb{R}}$ and the idea is to decompose r_3 according to the decomposition of a certain regular representation of $G_{\mathbb{R}}$ itself. The result is that one attaches to (most) representations of $G_{\mathbb{R}}$ a representation of $\overline{G}_{\mathbb{R}}$ and thus one obtains an inverse to S_v. In particular, to the discrete series representation π_{k-1} of $G_{\mathbb{R}}$ corresponds the discrete series representation $\overline{\pi}_{k/2}$ of $\overline{G}_{\mathbb{R}}$. This result seems to be new. A classical version of the correspondence $\pi_{k-1} \to \overline{\pi}_{1/2}$ (using theta functions in place of Weil's representation) was first obtained by T. Shintani ([41]).

In Section 5 we begin the local theory for non-archimedean places by describing S_v for a wide class of irreducible representations (the non-supercuspidal representations). After attaching pairs of quasi-characters of F_v to such representations we define S_v again by squaring these characters and inducing up to G_v. We also analyze the class 1 representations and compute eigenvalues for the generalized Hecke algebra. These results are useful in showing that our map is consistent with Shimura and Kubota at the finite places as well, i.e., eigenvalues for the Hecke ring are preserved (cf. Theorem 5.13). In Subsection 5.5 we describe how the basic non-archimedean Weil representation r_1 decomposes. Together with the results of Subsection 4.4 this leads to the global correspondence D^1 alluded to above.

In Section 6 we attempt to tie together the local threads of

Sections 4 and 5. Our goal is a global description of the constituents of the space of automorphic (genuine) forms on $\widetilde{G}_{\mathbb{A}}$. Roughly speaking, the theory of Weil representations expounded in Section 2 is mixed locally with the theory of Eisenstein series sketched in Section 3. The result is a complete characterization in Subsection 6.1 of the discrete non-cuspidal spectrum of $L^2(G_F \backslash \widetilde{G}_{\mathbb{A}})$. In classical terms the implication is that any square-integrable modular form of half-integral weight which is not a cusp form must be a "translate" of the basic theta-function $\theta(z)$. Since such forms are also shown to be generated by residues of Eisenstein series, this amounts to a kind of Siegel-Weil formula (cf. [48] and [10]).

In Subsection 6.2 we treat the cuspidal spectrum of $L^2(G_F \backslash \widetilde{G}_{\mathbb{A}})$. Here ignorance essentially dominates the situation. Nevertheless it is shown that all "non-trivial" global constituents of r_1 indeed contribute to the space of cusp forms. Similar results are discussed for r_3. Subsection 6.3 describes some speculations and problems which we hope to return to in future papers.

§2. Metaplectic Groups and Representations.

In [21] Kubota constructs a non-trivial two-fold covering group of $GL_2(\mathbb{A})$ over a totally imaginary number field. In this Section I shall describe the basic properties of such a group over an arbitrary number field and complete some details of Kubota's construction at the same time. I shall also recall Weil's construction in a form suitable for our purposes.

I start by discussing the local theory and first collect some elementary facts about topological group extensions.

Let G denote a group and T a subgroup of the torus regarded as a trivial G-space. A two-cocycle (or multiplier, or factor set) on G is a map from $G \times G$ to T satisfying

$$\alpha(g_1g_2, g_3)\alpha(g_1, g_2) = \alpha(g_1, g_2g_3)\alpha(g_2, g_3) \tag{2.1}$$

and

$$\alpha(g, e) = \alpha(e, g) = 1 \tag{2.2}$$

for all g, g_i in G. In addition, if G is locally compact, α will be called Borel if it is Borel measurable.

Following Moore [28], let $Z^2(G,T)$ denote the group of Borel 2-cocycles on G and let $B^2(G,T)$ denote its subgroup of "trivial" cocycles (cocycles of the form $s(g_1)s(g_2)s(g_1g_2)^{-1}$ with s a map from G to T). Then the quotient group $H^2(G,T)$ (the two-dimensional cohomology group of G with coefficients in T) represents equivalence classes of topological coverings groups of G by T which are central as group extensions.

To see this, let α be a representative of the cohomology class $\bar{a}$ in $H^2(G,T)$. Form the Borel space $G \times T$ and define multiplication in $G \times T$ by

$$\{g_1, \zeta_1\}\{g_2, \zeta_2\} = \{g_1g_2, \alpha(g_1, g_2)\zeta_1\zeta_2\} \quad . \tag{2.3}$$

One can check that $G \times T$ is a standard Borel group and that the product of Haar measures on G and T is an invariant measure for $G \times T$. Thus by [25] $G \times T = \overline{G}$ admits a unique locally compact topology compatible with the given Borel structure.

Note that the natural maps from T to $\overline{G}$ and $\overline{G}$ to G are continuous. (They are homomorphisms, and obviously Borel, hence they are automatically continuous.) The latter map, moreover, induces a homeomorphism of $\overline{G}/T$ with G. Thus we have an exact sequence of locally compact groups

$$1 \to T \to \overline{G} \to G \to 1 .$$

This sequence is central as a group extension and depends only on the equivalence class of α. Its natural Borel cross-section is $\ell: g \to (g,1)$.

2.1. Local Theory.

Let F denote a local field of zero characteristic. If F is archimedean, F is $\mathbb{R}$ or $\mathbb{C}$; if F is non-archimedean, F is a finite algebraic extension of the p-adic field $\mathbb{Q}_p$.

If F is non-archimedean, let O denote the ring of integers of F, U its group of units, P its maximal prime ideal, and ω a generator of P. Let $q = |\omega|^{-1}$ denote the residual characteristic of F.

The local metaplectic group is defined by a two-cocycle on $GL_2(F)$ which involves the Hilbert or quadratic norm residue symbol of F.

The Hilbert symbol $(\cdot,\cdot)$ is a symmetric bilinear map from $F^x \times F^x$ to Z_2 which takes (x,y) to 1 iff x in F^x is a norm from $F(\sqrt{y})$. In particular, (x,y) is identically 1 if y is a square. Thus $(\cdot,\cdot)$ is trivial on $(F^x)^2 \times (F^x)^2$ for <u>every</u> F and trivial on $F^x \times F^x$ itself if $F = \mathbb{C}$.

Some properties of the Hilbert symbol which we shall repeatedly use throughout this paper are collected below.

<u>Proposition 2.1</u>. (i) For each F, $(\cdot,\cdot)$ is continuous,

$$(a,b) = (a,-ab) = (a,(1-a)b) , \tag{2.4}$$

and

$$(a,b) = (-ab,a+b) ; \tag{2.5}$$

(ii) If q is odd, (u,v) is identically 1 on $U \times U$;

(iii) If q is even, and v in U is such that $v \equiv 1(4)$, then (u,v) is identically 1 on U.

The proof of this Proposition can be gleaned from Section 63 of O'meara [31] and Chapter 12 of Artin-Tate [1].

Now suppose $s = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(F)$ and set $x(s)$ equal to c or d according if c is non-zero or not.

Theorem 2.2. The map $\alpha: SL_2(F) \times SL_2(F) \to Z_2$, defined by

$$\alpha(s_1,s_2) = (x(s_1),x(s_2))(-x(s_1)x(s_2),x(s_1s_2)) \quad , \tag{2.6}$$

is a Borel two-cocycle on $SL_2(F)$. Moreover, this cocycle is cohomologically trivial if and only if $F = \mathbb{C}$.

This Theorem is the main result of [20]. According to our preliminary remarks it determines an exact sequence of topological groups

$$1 \to Z_2 \to \overline{SL_2}(F) \to SL_2(F) \to 1$$

where $\overline{SL_2}(F)$ is realized as the group of pairs $\{s,\zeta\}$ with multiplication given by (2.3). The topology for $\overline{SL_2}(F)$, however, is not the product topology of $SL_2(F)$ and Z_2 unless $F = \mathbb{C}$. Indeed, suppose N is a neighborhood basis for the identity in $SL_2(F)$. Then a neighborhood basis for $\overline{SL_2}(F)$ is provided by the sets $(U,1)$ where $U \in N$ and $\alpha(U,U)$ is identically one.

Proposition 2.3. If $F \neq \mathbb{C}$, each non-trivial topological extension of $SL_2(F)$ by Z_2 is isomorphic to the group $\overline{SL_2}(F)$ just constructed.

Proof. If $F = \mathbb{R}$, each such extension is automatically a connected Lie group, hence isomorphic to "the" two-sheeted cover of $SL_2(F)$ obtained by factoring its universal cover by $2\mathbb{Z}$. Suppose, on the other hand, that F is non-archimedean. What has to be shown is that $H^2(SL_2(F),Z_2) = Z_2$. For this we appeal to a result of C. Moore's.

Let E_F denote the (finite cyclic) group of roots of unity in F . Consider the short exact sequence

$$1 \to Z_2 \to E_F \to E_F/Z_2 \to 1 \quad .$$

The corresponding long exact sequence of cohomology groups is

$$\cdots H^1(SL_2(F), E_F/Z_2) \to H^2(SL_2(F), Z_2) \to H^2(SL_2(F), E_F) \to$$

$$H^2(SL_2(F), E_F/Z_2) \to H^3(SL_2(F), Z_2) \to \cdots .$$

Recall that $H^1(SL_2(F), T) = \mathrm{Hom}(SL_2(F), T)$. Moreover $SL_2(F)$ equals its commutator subgroup. Therefore $H^1(SL_2(F), E_F/Z_2) = \{1\}$ and $H^2(SL_2(F), Z_2)$ imbeds as a subgroup of $H^2(SL_2(F), E_F)$. But from Theorem 10.3 of [29] it follows that $H^2(SL_2(F), E_F) = E_F$. Thus the desired conclusion follows from the fact that $H^2(SL_2(F), Z_2)$ is non-trivial and each of its elements obviously has order at most two. □

Remark 2.4. As already remarked, a non-trivial two-fold cover of $SL_2(F)$ for F a non-archimedean field seems first to have been constructed by Weil in [47]. His construction, which we shall recall in Subsection 2.3, is really an existence proof. His general theory leads first to an extension

$$1 \to T \to M_p(2) \to SL_2(F) \to 1 ,$$

where T is the torus and $M_p(2)$ is a group of unitary operators on $L^2(F)$. Then it is shown that $M_p(2)$ determines a non-trivial element of order two in $H^2(SL_2(F), T)$.

Remark 2.5. In [20] Kubota constructs n-fold covers of $SL_2(F)$. His idea is to replace Hilbert's symbol in (2.6) by the n-th power norm residue symbol in F (assuming F contains the n-th roots of unity). In [29] Moore treats similar questions for a wider range of classical p-adic linear groups.

Now we must extend α to

$$G = GL_2(F) .$$

In fact we shall describe a two-fold cover of G which is a trivial non-central extension of $SL_2(F)$ by $F^\times$, i.e. a semi-direct product

of these groups.

If $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ belongs to G, write $g = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \det(g) \end{smallmatrix}\right] p(g)$ where

$$p(g) = \begin{bmatrix} a & b \\ \frac{c}{\det(g)} & \frac{d}{\det(g)} \end{bmatrix} \in SL_2(F) \quad . \tag{2.7}$$

For g_1, g_2 in G, define

$$\alpha^*(g_1, g_2) = \alpha(p(g_1)^{\det(g_2)}, p(g_2))\, v(\det(g_2), p(g_1)) \tag{2.8}$$

where

$$s^y = \left[\begin{smallmatrix} 1 & 0 \\ 0 & y \end{smallmatrix}\right]^{-1} s \left[\begin{smallmatrix} 1 & 0 \\ 0 & y \end{smallmatrix}\right] \tag{2.9}$$

and

$$v(y,s) = \begin{cases} 1 & \text{if } c \neq 0 \\ (y,d) & \text{otherwise} \end{cases} \tag{2.10}$$

if $s = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Note that the restriction of α^* to $SL_2(F) \times SL_2(F)$ coincides with α.

Proposition 2.6. If $y \in F^x$, and $\bar{s} = \{s, \zeta\} \in \overline{SL_2}(F)$, put $\bar{s}^y$ equal to $\{s^y, \zeta v(y,s)\}$. Then $\bar{s} \to \bar{s}^y$ is an automorphism of $\overline{SL_2}(F)$, and the semi-direct product of $\overline{SL_2}(F)$ and F^x it determines is isomorphic to the covering group $\bar{G}$ of G determined by (2.10).

Proof. It suffices to prove that

$$\alpha(s_1, s_2) = \alpha(s_1^y, s_2^y)\, v(y, s_1)\, v(y, s_2)\, v(y, s_1 s_2)$$

and this is verified in Kubota [21] by direct computation.

Remark 2.7. We shall refer to $\bar{G}$ as "the" metaplectic group even though there are several (cohomologically distinct) ways to extend α to G. We shall also realize it as the set of pairs $\{g, \zeta\}$ with $g \in G$, $\zeta \in Z_2$, and multiplication described by $\{g_1, \zeta_1\}\{g_2, \zeta_2\} = \{g_1 g_2, \alpha^*(g_1, g_2)\zeta_1\zeta_2\}$.

Now let B, A, N, and K denote the usual subgroups of G. Thus

$$A = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \ a_1 \in F^{\times} \right\}, \qquad N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \ b \in F \right\}$$

$$B = NA = \left\{ \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix} \right\}$$

and K is the standard maximal compact subgroup of G ($U(2)$ if $F = \mathbb{C}$, $O(2)$ if $F = \mathbb{R}$, and $GL(2,0)$ otherwise).

If H is <u>any</u> subgroup of G, $\overline{H}$ will denote its complete inverse image in $\overline{G}$. Moreover, if $\overline{G}$ splits over H, then $\overline{H}$ is the direct product of Z_2 and some subgroup H' of $\overline{G}$ isomorphic to H. We shall denote H' by H even though H' need not be uniquely determined by H.

In general, it is important to know whether or not $\overline{G}$ splits over the subgroups listed above. In particular, the Proposition below is useful in constructing the global metaplectic group. (Recall that if x belongs to a non-archimedean field its <u>order</u> is defined by the equation $x = \omega^{\mathrm{ord}(x)} u$, $u \in U$.)

<u>Proposition 2.8</u>. Suppose $F \neq \mathbb{C}$ or $\mathbb{R}$ and N (as usual) is a positive integer divisible by 4. Then $\overline{G}$ splits over the compact group

$$K^N = \{ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in K: \ a \equiv 1, \ c \equiv 0 (\mathrm{mod}\ N) \}.$$

More precisely, for $g = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in G$, set

$$s([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = \begin{cases} (c, d \det(g)) & \text{if } cd \neq 0 \text{ and } \mathrm{ord}(c) \text{ is odd} \\ 1 & \text{otherwise} \end{cases} . \tag{2.11}$$

Then <u>for all $g_1, g_2 \in K^N$</u>,

$$\alpha^*(g_1, g_2) = s(g_1) s(g_2) s(g_1 g_2)^{-1} . \tag{2.12}$$

<u>Proof</u>. Theorem 2 of [21] asserts that (2.12) is valid for all g_1, g_2 in $\{ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in K: \ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \equiv 1_2 (\mathrm{mod}\ N) \}$. But careful inspection

of Kubota's proof reveals that the conditions $b \equiv 0 \pmod{N}$ and $d \equiv 1 \pmod{N}$ are superfluous. Indeed Kubota's proof is computational and the crucial observation which makes it work is the Lemma below. We include its proof since Kubota does not.

Lemma 2.9. Suppose $k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K^N$. Then

$$s(k) = \begin{cases} (c, d(\det k)) & \text{if } c \neq 0 \text{ and } c \notin U \\ 1 & \text{otherwise.} \end{cases} \tag{2.13}$$

Proof. Throughout this proof assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K^N$. In particular, $\det(k) = ad - bc$ belongs to U.

Suppose first that $c \neq 0$ and $c \notin U$. Then clearly $cd \neq 0$. Indeed $d = 0$ implies that $\det(k) = -bc$, a contradiction since $c \notin U$ implies $-bc \notin U$. Thus if order (c) is odd,

$$s(k) = (c, d(\det(k))$$

by definition.

On the other hand, if $\operatorname{ord}(c) = 2n$ (where $n \neq 0$ since $c \notin U$) it remains to prove that

$$(c, d(\det k)) = 1 .$$

But $d(\det k) = ad^2 - dbc$, so $k \in K^N$ implies that $d(\det k) \equiv 1 \pmod{4}$. Thus

$$(c, d(\det(k))) = (\omega^{2n} u, u') = (u, u') = 1$$

by (iii) of Proposition 2.1.

To complete the proof it suffices to verify that $c \neq 0$ and $c \notin U$ if $cd \neq 0$ and $\operatorname{ord}(c)$ is odd. This is obvious, however, since if $c \in U$, then $\operatorname{ord}(c) = 0$, a contradiction, since $\operatorname{ord}(c)$ must be odd. □

Note that when the residual characteristic of F is odd, $K^N = K = GL(2, O_F)$.

Definition 2.10. If $F = \mathbb{R}$ or $\mathbb{C}$, let $s(g)$ denote the function on G which is identically one. If F is non-archimedean, let $s(g)$ be as in (2.11), and in general, let $\beta(g_1, g_2)$ denote the factor set $\alpha^*(g_1, g_2)\, s(g_1)s(g_2)s(g_1g_2)$.

Obviously β determines a covering group of G isomorphic to $\overline{G}$. But according to Proposition 2.8, its restriction to $K^N \times K^N$ is identically one. Thus $\overline{K}^N$ is isomorphic to $K^N \times Z_2$, and K^N lifts as a subgroup of $\overline{G}$ via the map $k \to \{k, 1\}$. For this reason we shall henceforth deal exclusively with β.

Lemma 2.11. Suppose

$$g_i = \begin{bmatrix} \mu_i & x_i \\ 0 & \lambda_i \end{bmatrix} \in B, \quad i = 1, 2 \quad .$$

Then $\beta(g_1, g_2) = (\mu_1, \lambda_2)$.

Proof. Since

$$\begin{bmatrix} \mu_i & x_i \\ 0 & \lambda_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mu_i\lambda_i \end{bmatrix} \begin{bmatrix} \mu_i & x_i \\ 0 & \mu_i^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \det(g_i) \end{bmatrix} p(g_i) \quad ,$$

it follows that

$$\alpha^*(g_1, g_2) = \alpha\left(\begin{bmatrix} \mu_1 & x_1\mu_2\lambda_2 \\ 0 & \mu_1^{-1} \end{bmatrix}, \begin{bmatrix} \mu_2 & x_2 \\ 0 & \mu_2^{-1} \end{bmatrix}\right) v(\mu_2\lambda_2, \begin{bmatrix} \mu_1 & x_1 \\ 0 & \mu_1^{-1} \end{bmatrix})$$

$$= (\mu_1^{-1}, \mu_2^{-1})(-\mu_1^{-1}\mu_2^{-1}, \mu_1^{-1}\mu_2^{-1})(\mu_2\lambda_2, \mu_1^{-1}) \quad .$$

But using (2.4) together with the symmetry and bilinearity of Hilbert's symbol, this last expression is easily seen to equal (μ_1, λ_2). Thus the Lemma follows from the identity

(2.14) $$s([\begin{smallmatrix}\mu & x\\ 0 & \lambda\end{smallmatrix}]) = 1$$

valid for all $[\begin{smallmatrix}\mu & x\\ 0 & \lambda\end{smallmatrix}] \in B$. □

Corollary 2.12. Suppose $\gamma = [\begin{smallmatrix}\mu & 0\\ 0 & \lambda\end{smallmatrix}]$. Then $\overline{\gamma} = \{\gamma,\zeta\}$ is central in $\overline{A}$ iff γ is a square in A.

Proof. Suppose $\overline{\gamma}' = \{\gamma',\zeta'\} = \{[\begin{smallmatrix}\mu' & 0\\ 0 & \lambda'\end{smallmatrix}],\zeta'\}$ is arbitrary in $\overline{A}$. Then $\overline{\gamma}\,\overline{\gamma}' = \overline{\gamma}'\,\overline{\gamma}$ iff $\{\gamma\gamma',\beta(\gamma,\gamma')\zeta\zeta\} = \{\gamma'\gamma,\beta(\gamma',\gamma)\zeta\zeta'\}$ iff $\beta(\gamma,\gamma') = \beta(\gamma',\gamma)$ iff

(2.15) $$(\mu,\lambda') = (\mu',\lambda) .$$

So suppose first that γ is a square in A. Then both μ and λ are squares in F^x and (2.15) obtains by default (both sides equal one for all μ',λ').

On the other hand, if μ, say, is not a square in F^x, then by the non-triviality of Hilbert's symbol,

$$(\mu,\lambda') = -1 \quad \text{for some} \quad \lambda' \in F^x .$$

This means that (2.15) fails for $\mu' = 1$, say. Thus $\overline{\gamma}$ will not commute with $\{[\begin{smallmatrix}1 & 0\\ 0 & \lambda'\end{smallmatrix}],1\}$ and the proof is complete. □

Corollary 2.12'. The subgroup N of G lifts as a subgroup of $\overline{G}$. So does A^2 with

$$A^2 = \{\gamma \in A:\ \gamma = \delta^2,\ \delta \in A\} .$$

Proof. Obvious.

Corollary 2.13. Fix $F \neq \mathbb{C}$. Then:

(a) The center of $\overline{G}$ is

$$Z(\overline{G}) = \{\{[\begin{smallmatrix}z & 0\\ 0 & z\end{smallmatrix}],\zeta\}:\ z \in (F^x)^2\};$$

(b) Suppose

$$\overline{G}^2 = \{\{g,\zeta\} \in \overline{G}:\ \det(g) \in (F^x)^2\} .$$

Then the center of $\overline{G}^2$ is $\overline{Z} = \{\{[\begin{smallmatrix}z & 0\\ 0 & z\end{smallmatrix}],\zeta\}:\ z \in F^x\}$.

Proof. Since $\overline{g} = \{g,\zeta\} \in Z(\overline{G})$ only if $g \in Z(G)$, i.e. only if $g = \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$ with $z \in F^x$, it suffices to prove that $\{\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right],\zeta\}$ commutes with <u>every</u> $\overline{g}' = (\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right],\zeta') \in \overline{G}$ iff z is a square in F^x.

Clearly $\{\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right],\zeta\}$ commutes with $\overline{g}'$ iff $\beta(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right],g') = \beta(g',\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right])$. But a simple computation shows that

$$a^*(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right],g') = (z,c)$$

and

$$\alpha^*(g',\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]) = (z,c)(z,\det(g')) .$$

So since

$$s(g')^{-1}s(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right])^{-1}s(g'\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]) = s(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right])^{-1}s(g')^{-1}s(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]g'),$$

$\{\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right],\zeta\} \in Z(\overline{G})$ if and only if

$$(z,\det(g')) = 1 \tag{2.16}$$

<u>for all $g' \in G$</u>, or, since Hilbert's symbol is non-trivial, $\{\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right],\zeta\} \in Z(\overline{G})$ iff z is a square. On the other hand, if $\det(g') \in (F^x)^2$, (2.16) holds for <u>all</u> z, and hence (b) too is established. □

2.2. Global Theory.

In this paragraph, F will denote an arbitrary number field, v a place of F, F_v the completion of F at v, and $\mathbb{A}$ the adeles of F. Thus the G of Section 2.1 becomes $G_v = GL_2(F_v)$, its maximal compact subgroup is K_v, A is A_v, etc. Our interest henceforth is in the global group

$$G_{\mathbb{A}} = GL_2(\mathbb{A})$$

and its two fold cover $\overline{G}_{\mathbb{A}}$.

If g and g' are arbitrary in $G_{\mathbb{A}}$, put $\beta_v(g,g') = \beta(g_v,g'_v)$, if $g = (g_v)$ and $g' = (g'_v)$. According to Proposition 2.8, $\beta_v(g,g') = 1$ for almost every v. Thus it makes sense to define $\beta_{\mathbb{A}}$ on $G_{\mathbb{A}} \times G_{\mathbb{A}}$ by

$$\beta_{\mathbb{A}}(g,g') = \prod_v \beta_v(g,g') \tag{2.17}$$

the product extending over all the primes of F. Since $\beta_{\mathbb{A}}$ is obviously a Borel factor set on $G_{\mathbb{A}}$ it determines an extension of $G_{\mathbb{A}}$ which we shall denote by $\overline{G}_{\mathbb{A}}$ and realize as the set of pairs $\{g,\zeta\}$, $g \in G_{\mathbb{A}}$, $\zeta \in Z_2$, with group multiplication given by

$$\{g_1,\zeta_1\}\{g_2,\zeta_2\} = \{g_1g_2,\beta_{\mathbb{A}}(g_1,g_2)\zeta_1\zeta_2\}.$$

Important subgroups of $G_{\mathbb{A}}$ include

$$K_0^N = \prod_{v<\infty} K_v^N$$

and

$$G_F = GL_2(F).$$

Proposition 2.14. The subgroup K_0^N of $G_{\mathbb{A}}$ lifts to a subgroup of $\overline{G}_{\mathbb{A}}$ via the map $k_0 \to \{k_0,1\}$.

Proof. Proposition 2.8.

Proposition 2.15. For each $\gamma \in G_F$, let

$$s_{\mathbb{A}}(\gamma) = \prod_v s_v(\gamma)$$

the product extending over all (finite) primes v of F. Then the map

$$\gamma \to \{\gamma, s_{\mathbb{A}}(\gamma)\}$$

provides an isomorphism between G_F and a subgroup of $\overline{G}_{\mathbb{A}}$.

Proof. Note first that if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_F$, the v-order of c is 0 for almost every v. Thus $s_v(\gamma) = 1$ for almost every v and the product appearing in (2.18) is finite, i.e. $s_{\mathbb{A}}(\gamma)$ is well-defined. To prove the Proposition it will suffice to prove that

$$s_{\mathbb{A}}(\gamma)s_{\mathbb{A}}(\gamma')\beta_{\mathbb{A}}(\gamma,\gamma') = s_{\mathbb{A}}(\gamma\gamma') \tag{2.18}$$

for all γ, γ' in G_F.

So fix γ and γ' in G_F. For almost all v, all the entries of γ and γ' will be units. Thus $\alpha_v(\gamma,\gamma') = 1$ for almost all v. (By Proposition 2.1, Hilbert's symbol is trivial on units if v is finite and the residual characteristic of F_v is odd.) On the other hand, for all v,

$$\alpha_v(\gamma,\gamma') = (r_1,r_2)_v(r_3,r_4)_v$$

with $r_1, r_2, r_3, r_4 \in F$. Consequently, by the product formula for Hilbert's symbol (quadratic reciprocity for F),

$$\prod_v \alpha_v(\gamma,\gamma') = 1 .$$

That is,

$$\beta_{\mathbb{A}}(\gamma,\gamma') = s_{\mathbb{A}}(\gamma)s_{\mathbb{A}}(\gamma')s_{\mathbb{A}}(\gamma\gamma')$$

as was to be shown. □

Because of this Proposition we can make sense now out of the homogeneous space

$$\overline{X} = Z_\infty^0 G_F \backslash \overline{G}_{\mathbb{A}} \tag{2.19}$$

where Z_∞^0 denotes the subgroup of the center of G_∞ consisting of positive real matrices $\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$. (Cf. Corollaries 2.12' and 2.13.) This space will be the focus of our attention in Section 3.

The Proposition below makes it possible to relate functions on $\overline{X}$ with classical forms defined in the upper half-plane. Before stating it we collect some facts concerning the quadratic power residue symbol.

Suppose $a,b \in F^x$ with b relatively prime to a and 2. Let S_∞ denote the set of archimedean places of F. Suppose that

$$(b) = \prod_v v^{\mathrm{ord}_v(b)}$$

where (b) denotes the F-ideal generated by b and the product extends over all prime ideals of O_F (the ring of integers of F). Only finitely many v will be such that $\mathrm{ord}_v(b) \neq 0$ and each such v will be relatively prime to 2 and a.

The quadratic power residue symbol $(\frac{a}{v})$ is then 1 if $x^2 = a$ has a solution in O_v and -1 otherwise. The quadratic power residue symbol $(\frac{a}{b})$ is

$$(\frac{a}{b}) = \prod_{\substack{v \notin S_\infty \\ v \nmid 2}} (\frac{a}{v})^{\mathrm{ord}_v(b)}$$

Now consider the congruence subgroup

$$\Gamma_1(N) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,O): a \equiv d \equiv 1,\ c \equiv 0(N)\}. \tag{2.20}$$

Clearly

$$\Gamma_1(N) = G_F \cap G_\infty^O K_O^N \tag{2.21}$$

where $G_\infty^O = \prod_{v \in S_\infty} G_v^O$, the product of the connected components of the archimedean completions of G. Note that for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL(2,O)$, d is relatively prime to c and 2.

Proposition 2.16. If d is relatively prime to c and 2, let

$$(\frac{c}{d})_s = (\frac{c}{d}) \prod_{v \in S_\infty} (c,d)_v \tag{2.22}$$

If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to $\Gamma_1(N)$, put

$$\chi(\gamma) = \begin{cases} (\frac{c}{d})_s & \text{if } c \neq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.23}$$

Then

$$(2.24) \qquad s_{\mathbb{A}}(\gamma) = \chi(\gamma)$$

for $\gamma \in \Gamma_1(N)$.

Proof. From the definition of everything in sight it follows that

$$s_{\mathbb{A}}(\gamma) = \prod_v s_v(\gamma) = \prod_{v \text{ finite}} s_v(\gamma)$$

Thus the Proposition is obvious if $c = 0$ (since both sides of (2.24) are then one).

Assume now that $c \neq 0$ and recall that c and d are relatively prime. By Lemma 2.9,

$$s_v(\gamma) = \begin{cases} (c, d, \det(\gamma))_v & \text{if } c \neq 0 \text{ and } c \notin U_v \\ 1 & \text{otherwise} \end{cases}$$

Therefore

$$s_v(\gamma) = \begin{cases} (c, d(\det \gamma)_v & \text{if } v|c \\ 1 & \text{otherwise} \end{cases}$$

and consequently

$$s_{\mathbb{A}}(\gamma) = \prod_{v|c} (c, d(\det \gamma))_v.$$

By assumption, $\det(\gamma)$ is a unit for <u>each</u> v. Moreover, $\det(\gamma) \equiv 1(4)$. Thus by Proposition 2.1, and the product fromula for Hilbert's symbol

$$\prod_{v|c} (c, \det \gamma)_v = \prod_{v \nmid c} (c, \det(\gamma)) \prod_{v \in S_\infty} (c, \det \gamma) = 1.$$

Consequently $\prod_{v|c} (c, d(\det \gamma) = \prod_{v|c} (c, d)_v$, which means it suffices to prove

$$(2.25) \qquad \prod_{v|c} (c, d)_v = \left(\frac{c}{d}\right)_s.$$

To simplify matters, assume F has class number one. (The Proposition is undoubtedly true without this assumption.) To prove (2.25) recall the following basic formulas ([14], Chapter 13): for each prime integer w in F dividing both c and 2,

$$(\frac{w}{d}) = \prod_{\substack{v\in S_\infty \\ v|2}} (w,d)_v \tag{2.26}$$

(Supplementary Reciprocity Law); for each prime w which divides c but not 2,

$$(\frac{w}{d}) = (w,d)_w \prod_{\substack{v\in S_\infty \\ v|2}} (w,d)_v \tag{2.27}$$

(Power reciprocity formula). Note that if d is a v-unit for odd v dividing c, $(w,d)_v = 1$. Consequently (2.26) can be rewritten as

$$(\frac{w}{d}) = \prod_{\substack{v|c \\ v\nmid 2}} (w,d)_v \prod_{\substack{v\in S_\infty \\ v|2}} (w,d)_v \,. \tag{2.28}$$

Now suppose $c = \prod_{w|c} w^{\mathrm{ord}_w(c)}$. By the multiplicativity of the power residue symbol, (2.28) and (2.27) can be multiplied for each $w|c$ to obtain

$$(\frac{c}{d}) = (\prod_{v|c} (c,d)_v) \prod_{v\in S_\infty} (c,d)_v \,.$$

Thus the proof is complete. (Units play no role since $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1(N)$ implies $(u,d)_v = 1$.) □

Two special cases of Proposition 2.16 will be of particular interest in Section 3. The first assume F is a <u>totally imaginary</u> number field, in which case

$$\chi(\gamma) = \begin{cases} (\frac{c}{d}) & \text{if } c \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.29}$$

since $(c,d)_\infty$ is identically one.

Corollary 2.17. (Cf. Kubota [19], cf. Theorem 6.1 of Bass-Milnor-Serre [2]). If F is totally imaginary, $\chi(\gamma)$ defines a character of $\Gamma_1(N)$, i.e. $\chi(\gamma_1\gamma_2) = \chi(\gamma_1)\chi(\gamma_2)$ for all $\gamma_1,\gamma_2 \in \Gamma_1(N)$.

Proof. Since F is totally imaginary, $\beta_{\mathbb{A}}$ is identically one on $G_\infty^0 K_0^N$ in $G_{\mathbb{A}}$. So suppose γ_1 and γ_2 are arbitrary in $\Gamma_1(N)$. Since γ_1 and γ_2 belong to G_F,

$$s_{\mathbb{A}}(\gamma_1\gamma_2) = \beta_{\mathbb{A}}(\gamma_1,\gamma_2)s_{\mathbb{A}}(\gamma_1)s_{\mathbb{A}}(\gamma_2)$$

(by (2.18)). But $\beta_{\mathbb{A}}(\gamma_1,\gamma_2) = 1$. Thus the Corollary is immediate from Proposition 2.16. □

The existence of such a character (with the assumption that F is totally imaginary) provides a starting point for Kubota's recent investiations ([21] through [23]). Its relation to the congruence subgroup problem is discussed in [2].

The second example I have in mind assumes $F = Q$. In this case,

$$\chi(\gamma) = \begin{cases} \left(\frac{d}{|d|}\right) & \text{if } c \neq 0,\ c > 0 \text{ or } d > 0 \\ -\left(\frac{c}{|d|}\right) & \text{if } c \neq 0,\ c < 0 \text{ and } d < 0 \\ 1 & \text{if } c = 0. \end{cases}$$

Now χ no longer defines a character of $\Gamma_1(N)$ ($\overline{G}_\infty$ is not a trivial extension of G_∞). However $\chi(\gamma)$ is still a "multiplier system" for $\Gamma_1(N)$. More precisely, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, let $J^*(\gamma,z) = (cz+d)^{1/2}$. Here $w^{1/2}$ is chosen so that $-\pi/2 < \arg(w^{1/2}) \leq \pi/2$.

Then $|\chi(\gamma)| = 1$ and

$$\frac{\chi(\gamma_1\gamma_2)}{\chi(\gamma_1)\chi(\gamma_2)} = \frac{J^*(\gamma_1,\gamma_2 z)J^*(\gamma_2,z)}{J^*(\gamma_1\gamma_2,z)} \tag{2.31}$$

for all $\gamma \in \Gamma_1(N)$. I.e., $\chi(\gamma)$ is a multiplier system for $\Gamma_1(N)$ of dimension 1/2.

2.3. Weil's Metaplectic Representation.

In this Section I want to sketch Weil's general theory of the metaplectic representation and reformulate parts of it in a form suitable for the construction of automorphic forms on the metaplectic group.

Roughly speaking, to each abstract symplectic group, one associates a projective representation (Weil's metaplectic representation). This representation operates in L^2 of the group and its associated multiplier is of order two. Thus Weil's representation reproduces a two fold covering group which repreduces $\overline{SL}_2(F)$ when the underlying group is $SL(2)$.

We begin by recalling some basic properties of projective representations and the group extensions they determine.

If H is a Hilbert space, the torus imbeds in the obvious way as a central subgroup of $U(H)$ equipped with the strong operator topology. A projective (unitary) representation of G in H is then a continuous homomorphism π^* from G to $U(H)/T$.

Our interest will be in the cocycle (or multiplier) representations of G canonically associated to such projective representations π^*. These are obtained by choosing a Borel cross-section f of $U(H)/T$ in $U(H)$ and introducing the map $\pi = f \circ \pi^*$. If $f(T) = 1$, this map from G to $U(H)$ must automatically be Borel and satisfy

$$\pi(e) = I \tag{2.32}$$

and

$$\pi(g_1)\pi(g_2) = \alpha(g_1, g_2)\pi(g_1, g_2) \quad . \tag{2.33}$$

for all $g_1, g_2 \in G$. Here α is a Borel function from $G \times G$ to T which (by the associative law in G) must be a two-cocycle on G with values in T.

Note that α above obviously depends on the choice of cross-section f. However, if f' is another such cross-section,

the cocycle it determines will be cohomologous to α. Thus each projective representation uniquely determines an element of $H^2(G,T)$.

Any Borel map from G to $U(H)$ satisfying (2.32) and (2.33) is called a multiplier representation with multiplier α (or, simply, an α-representation) and all such representation arise from projective representations as above. More precisely, if π is any α-representation, set $\pi^* = p\circ\pi$, where p is the natural projection from $U(H)$ to $U(H)/T$. Then $\pi = f\circ\pi^*$, and α lies in the cohomology class determined by π^*. In this sense, each projective representation is essentially a family of multiplier representations with cohomologous multipliers.

Throughout this paper we shall want to distinguish between representations of the metapletic group which factor through Z_2 and those that do not. Moreover, we shall often want to confuse those that don't with cocycle representations of $GL(2)$.

In general, if $\overline{G}$ is an extension of the locally compact group G by some subgroup T of the torus we shall call π genuine if $\pi(t) = t\circ I$ for all $t \in T \subset G$. For the sake of exposition we assume below that multiplication in $\overline{G}$ is determined by some fixed cocycle α.

Proposition 2.18. Let $\iota : G \to \overline{G}$ denote the (Borel) cross-section $\iota(g) = (g,1)$ and suppose π is a genuine representation of $\overline{G}$. Then:

(a) The map $\pi\circ\iota$ from G to $U(H)$ is an α-representation of G;

(b) The correspondence $\pi \to \pi\circ\iota$ is a bijection between the collection of genuine representations of $\overline{G}$ and α-representations of G;

(c) This correspondence preserves unitary equivalence and direct sums.

Proof. Part (a) follows immediately from the fact that

$\ell(g_1g_2) = \alpha(g_1,g_2)\ell(g_1)\ell(g_2)$. To prove (b), suppose π' is an α-representation of G in H and define

$$\pi(t\ell(g)) = t\pi'(g).$$

Then π is a Borel homomorphism of $\overline{G}$ into $U(H)$, whence continuous, and obviously "genuine". The rest of the proposition follows from the fact that an operator A intertwines the α-representations π_1' and π_2' if and only if it intertwines the corresponding ordinary representations π_1 and π_2. □

We shall now describe Weil's construction in earnest.

The symplectic groups of Weil's general theory are attached to locally compact abelian groups G so the notation $\langle x,x^*\rangle$ will denote the value of the character x^* in G^* (the dual group to G) at $x \in G$. Since Weil's theory is essentially empty unless G is isomorphic to G^* we shall assume throughout that this is the case.

Example 2.19. Let denote a local field of characteristic zero and V a finite-dimensional vector space defined over F. Fix q to be a non-degenerate quadratic form on V and τ the canonical non-trivial additive character of F described in [46]. The identity

$$\langle X,Y\rangle = \tau(q(X,Y)), \quad (X,Y \in V)$$

where

$$q(X,Y) = q(X+Y)-q(X)-q(Y),$$

establishes a self-duality for the additive group of V equipped with its obvious topology. Thus Weil's theory will be applicable in particular to $G = V$ together with (q,τ). This example in fact will suffice for the applications we have in mind.

For each $w = (u,u^*)$ in $G \times G^*$ let $U'(w)$ denote the unitary operator in $L^2(G)$ defined by

$$U'(w)\phi(x) = \langle x,u^*\rangle\phi(x+u).$$

Then $U'(w_1)U'(w_2) = F(w_1,w_2)U'(w_1+w_2)$ where

$$F(w_1,w_2) = \langle u_1, u_2^* \rangle$$

if $w_i = (u_i,u_i^*)$. That is, U' is a multiplier representation of $G \times G^*$ with multiplier $1/F$ and the family of operators

$$U(w,t) = tU'(w) \quad (w \in G \times G,\ t \in T)$$

comprises a group with composition law given by

$$(2.34) \qquad (w_1,t_1)(w_2,t_2) = (w_1+w_2, F(w_1,w_2)t_1t_2).$$

In particular, the multiplier representation U' determines an extension of $G \times G^*$ by T which is called the Heisenberg group of G and denoted by $A(G)$. In case $G = \mathbb{R}$, $A(G)$ is (the exponentiation of) the familiar Heisenberg group

$$\left\{ \begin{bmatrix} 1 & x & t \\ 0 & 1 & x^* \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

In general, T is the center of $A(G)$; $A(G)$ may be viewed either as $G \times G^* \times T$ equipped with the group law (2.34) or as the group of operators $\{U(w,t)\}$ (in which case it is denoted by $\overline{A}(G)$). The crucial fact is that $U(w,t)$ is an irreducible representation; in fact, $U(w,t)$ is the unique irreducible representation of $A(G)$ which leaves T pointwise fixed. Therefore at least the first part of the result below is plausible.

Theorem 2.20 (Segal). Let $B_0(G)$ denote the group of automorphisms of $A(G)$ leaving T pointwise fixed, $\overline{B_0(G)}$ the normalizer of $\overline{A(G)}$ in $L^2(G)$, and let

$$p_0 : \overline{B_0(G)} \to B_0(G)$$

be the natural projection. Then p_0 is onto with kernel T.

In other words, there exists a multiplier representation

of $B_0(G)$ in $L^2(G)$ whose range (choosing the constant in all possible ways) coincides with $\overline{B_0(G)}$. Indeed if $s \in B_0(G)$, the formula $U^s(w,t) = U((w,t))s)$ defines an irreducible unitary representation of $A(G)$ which fixes $A(G)$ pointwise and hence is equivalent to U. Therefore $U((w,t)s) = r^{-1}(s)U(w,t)r(s)$ for some unitary operator $r(s)$ in $L^2(G)$ (uniquely determined up to a scalar in T) and

$$s \to r(s)$$

determines a multiplier representation of $B_0(G)$ (the Weil representation) satisfying $p_0(r(s)) = s$.

This "abstract" Weil representation is relevant to $SL(2)$ since $B_0(G)$ is the semi-direct product of $(G \times G^*)^*$ and an abstract symplectic group which reduces to $SL_2(F)$ when $G = F$.

More precisely, since s in $B_0(G)$ fixes $(1,t)$, it is completely determined by its restriction to $G \times G^* \times \{1\}$ where it is of the form $(w,1) \overset{s}{\to} (w\sigma, f(w))$. Thus on $G \times G^* \times T$ it is of the form (σ,f) where

$$(w,t)s = (w,t)(\sigma,f) = (w\sigma, f(w)t),$$

σ is an automorphism of $G \times G^*$, $f: G \times G^* \to T$ is continuous, and

$$\frac{F(w_1\sigma, w_2\sigma)}{F(w_1,w_2)} = \frac{f(w_1+w_2)}{f(w_1)f(w_2)} \tag{2.35}$$

Conversely, each pair (σ,f) satisfying (2.35) defines an automorphism of $A(G)$ fixing T pointwise, so $B_0(G)=\{(\sigma,f)\}$, with group law

$$(\sigma,f)(\sigma',f') = (\sigma\sigma', f'')$$

if $f''(w) = f(w)f'(w\sigma)$.

Now let $Sp(G)$ denote the abstract symplectic group of automorphisms of $G \times G^*$ which leave invariant the bicharacter

$$\frac{F(w_1,w_2)}{F(w_2,w_1)} = \frac{\langle x_1,x_2^*\rangle}{\langle x_2,x_1^*\rangle} \qquad (w_i = (x_i,x_i^*))$$

Using (2.35) one checks that if $(\sigma,f) = s \in B_0(G)$ then $\sigma \in Sp(G)$. Our claim was that the exact sequence

$$(2.36) \qquad 1 \to (G \times G^*)^* \to B_0(G) \to Sp(G) \to 1$$

actually splits.

To check this it is convenient to describe $Sp(G)$ in matrix form. Since each $\sigma \in Aut(G \times G^*)$ is of the form

$$(x,x^*) \to (x,x^*)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

where $\alpha: G \to G$, $\beta: G \to G^*$, $\gamma: G^* \to G^*$, we shall, following Weil, write

$$\sigma = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix},$$

and define

$$\sigma^I = \begin{bmatrix}\delta^* & -\beta^*\\ -\gamma^* & \alpha^*\end{bmatrix},$$

where α^* denotes the appropriate map dual to α. Then $\sigma \in Aut(G \times G^*)$ is symplectic iff $\sigma\sigma^I = I$.

Given $\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \sigma \in Sp(G)$, define f_σ on $G \times G^*$ by

$$f_\sigma(u,u^*) = \langle u, 2^{-1}u\alpha\beta^*\rangle\langle 2^{-1}u^*\gamma\delta^*, u^*\rangle\langle u^*\gamma, u\beta\rangle .$$

Here we are assuming, as we do throughout, that $x \mapsto 2x$ is an automorphism of G. Then f_σ and σ satisfy (2.35) and the map

$$\sigma \to (\sigma,f)$$

is a monomorphism of $sp(G)$ into $B_0(G)$ which splits (2.36).

In short, Theorem 2.20 produces an exact sequence

$$(2.37) \qquad 1 \to T \to \overline{B_0(G)} \to B_0(G) \to 1$$

and $B_0(G)$ contains a copy of $Sp(G)$.

Example 2.21. Suppose F is local and $G = (V,q,\tau)$ is as in Example 2.19. Then $SL_2(F)$ can be imbedded homomorphically in $Sp(G)$ by allowing each element of F to act on V via scalar multiplication. (Since $\alpha^* = a$ for each $\alpha \in F$, $\sigma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL(2,F)$ obviously satisfies $\sigma\sigma^I = I$; note that $Sp(G) = SL_2(F)$ if $G = V$ but in all other cases properly contains it.) From this it follows that one can associate to each quadratic form (q,V) a natural projective representation of $SL(F)$ in $L^2(V)$ which we shall call the Weil representation attached to (q,V) and denote by r_q.

Before further analyzing this representation we need to adapt Weil's general theory to the special context of Examples 2.19 and 2.21. Recall that V is a vector space over F and τ is a non-trivial additive character of F (fixed once and for all). If X^* belongs to the linear dual V^* we shall denote its value at X in V by $[X,X^*]$. The natural isomorphism between V and V^* is then $[X,Y] = q(X,Y)$.

Using $[\cdot,\cdot]$ in place of $\langle\cdot,\cdot\rangle$ one can now "linearize" the theory just sketched, introducing:

(a) the bilinear form $B(z_1,z_2) = [X_1,X_2^*]$ in place of the bicharacter $F(W_1,W_2)$;

(b) the Heisenberg group $A(V)$ in place of $A(G)$;

(c) the symplectic group $Sp(V)$ in place of $Sp(G)$; and

(d) the pseudo-symplectic group $Ps(V)$ in place of $B_0(G)$.

A typical element of $Ps(V)$ is of the form (σ,f) where $\sigma \in Aut(V\times V^*)$ belongs to $Sp(V)$ and $f: V \times V^* \to F$ satisfies

$$f(z_1+z_2)-f(z_1)-f(z_2) = B(z_1\sigma,z_2\sigma)-B(z_1,z_2). \tag{2.38}$$

This relation, of course, is the linearlization of (2.35): taking τ of both sides of it yields (2.35). Note that $Ps(V) \neq B_0(G)$. In fact $Ps(V)$ is the subgroup of $B_0(G)$ consisting of pairs (σ,f) with f quadratic; cf. (2.39) below.

Note now that (2.38) associates to each σ in $Sp(V)$ one and only one quadratic form on $V \times V^*$. (We are assuming F

has characteristic zero, in particular, characteristic not equal to two.) Therefore the homomorphism

$$(\sigma, f) \to f$$

is an <u>isomorphism</u> between $Ps(V)$ and $Sp(V)$.

Since the same map from $B_0(G)$ to $Sp(G)$ has kernel isomorphic to $(G\times G^*)^*$ the analogy between the exponentiated and unexponentiated theories breaks down here. However, the map

$$(\sigma, f) \xrightarrow{\mu} (\sigma, \tau \circ f) \tag{2.39}$$

does <u>imbed</u> $Ps(V)$ homomorphically in $B_0(G)$ so we can still obtain an extension of $Ps(V)$ by pulling back (2.37) through μ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & M_p(V) & \xrightarrow{p} & Sp(V) & \longrightarrow & 1 \\ & & & & & & \downarrow \mu & & \\ 1 & \longrightarrow & T & \longrightarrow & \overline{B_0(G)} & \xrightarrow{p_0} & B_0(G) & \longrightarrow & 1 \end{array}$$

The group

$$Mp(V) = \{(s, \overline{s}) \in P_s(V) \times \overline{B_0(G)} : \; p_0(\overline{s}) = \mu(s)\}$$

is Weil's general metaplectic group. It is a central extension of $Sp(V)$ by T .

Weil's results concerning the non-triviality of $Mp(V)$ include the following:

(a) $Mp(V)$ always reduces to an extension of $Sp(V)$ by Z_2, i.e. the cohomology class it determines in $H^2(G,T)$ is of order 2 ;

(b) the extension $Mp(V)$ is in general non-trivial; in particular, if $F \neq \mathbb{C}$, and V is one-dimensional, $Mp(V)$ is always a non-trivial cover of $Sp(V)$ by Z_2 .

These results yield a (topological) central extension

$$1 \to Z_2 \to \overline{Sp}(V) \to Sp(V) \to 1 \tag{2.40}$$

which by Proposition 2.3 must coincide with

$$1 \to Z_2 \to \overline{SL}_2(F) \to SL_2(F) \to 1$$

when $V = F$. I.e., Weil's metaplectic group generalizes the metaplectic cover of $SL_2(F)$ constructed in 2.2.

Although Weil's construction does not immediately yield an explicit factor set for $\overline{SL}_2(F)$ it does a priori realize this group as a group of operators. In fact, it provides a host of representations for this group. These correspond to quadratic forms over F and in explicit terms are realized as follows.

Fix a Haar measure $d_\tau y$ on F such that the Fourier transform

$$\hat{f}(x) = \int f(y)\tau(2xy)\,d_\tau y$$

satisfies $(\hat{f})\hat{}\,(x) = f(-x)$. Recall that τ is the canonical character of F whose conductor is O_F. Given (q,V), fix an orthogonal basis $X_1,\dots,X_n$ of V such that if $X = \Sigma x_i X_i$, then $q(X) = q(x_1,\dots,x_n) = \Sigma \alpha_i x_i^2$, with $\alpha_i = \frac{1}{2} q(X_i,X_i)$. Let τ_i denote the character $\tau_i(y) = \tau(\alpha_i y)$ of F, $i = 1,\dots,n$, and let $d_i y$ denote the corresponding Haar measure on F, normalized as above.

If F is non-archimedean, the limit

$$\gamma(\tau_i) = \lim_{m\to\infty} \int_{P^m} \tau_i(y^2)\,d_i y$$

is known to exist (by Weil [47] it is an eighth root of unity). Consequently the invariant

$$\gamma(q,\tau) = \prod_{i=1}^{n} \gamma(\tau_i)$$

is well-defined. If $F = \mathbb{R}$, and $\tau_i(x) = \exp(\pi\sqrt{-1}\,\alpha_i x)$, set $\gamma(\tau_i)$ equal to $\exp(\frac{\pi}{4}\sqrt{-1}\,\operatorname{sgn}(\alpha_i))$ and define $\gamma(q,\tau)$ as above. Finally, if $F = \mathbb{C}$, set $\gamma(q,\tau)$ identically equal to 1. The Fourier transform on $L^2(V)$ is defined to be

$$\hat{\Phi}(X) = \int_V \Phi(Y)\tau(q(X,Y))dY$$

where $dY = \prod_{i=1}^{n} d_i y$.

Theorem 2.22. For each $t \in F^x$ let τ_t denote the character of F defined by $\tau_t(x) = \tau(tx)$ and denote $\gamma(q,\tau_t)$ by $\gamma(q,t)$. Suppose we use (q,V,τ) to imbed $SL_2(F)$ in $Sp(V)$ (and $B_0(V)$). Then a cross-section of $\overline{Sp}(V)$ over $SL_2(F)$ (c.f. (2.40)) is provided by the maps

$$(2.41) \qquad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \to r(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix})\Phi(X) = \tau_t(bq(X))\Phi(X)$$

$$(2.42) \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \to r(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})\Phi(X) = \gamma(q,t)^{-1}\hat{\Phi}(-X)$$

($\Phi \in L^2(V)$). More precisely, for each $t \in F^x$, these maps extend to a multiplier representation of $SL_2(F)$ in $L^2(V)$ whose associated cocycle is of order two.

Remark 2.23. In (2.42) the Fourier transform is taken with respect to τ_t and Haar measure $d_tY = |t|^{n/2}dY$. Thus

$$\hat{\Phi}(X) = \int_V \Phi(Y)\tau_t(q(X,Y))|t|^{n/2}dY .$$

Proof of Theorem 2.22. Without loss of generality we assume $t = 1$. Since $dY = \prod_{t=1}^{n} d_i Y$ it is easy to check that the operators in question are tensor products of the operators in $L^2(F)$ corresponding to τ_i, namely

$$r(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix})f(x) = \tau_i(bx^2)f(x)$$

and

$$r(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})f(x) = \gamma(\tau_i)^{-1}\hat{f}(-x) ,$$

$i = 1,\ldots,n$. Thus the Theorem is reduced to the case $V = F$, $q(x) = x^2$, precisely Theorem 1.4.1 of Shalika [35].

We note that $SL_2(F)$ is generated by elements of the form

$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ subject to certain relations. Thus one could follow Shalika directly and prove Theorem 2.22 by checking that these relations are preserved by the operators $r(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix})$ and $r(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})$. In any case, the resulting multiplier representation of $SL_2(F)$ will be denoted r_q and called "the" Weil representation attached to the quadratic form q. (This is an explicit realization of the representation r_q of Example 2.21).

Corollary 2.24. r_q is ordinary if and only if V is even dimensional.

We conclude this Section with some remarks. Although the group of operators $\overline{B}_0(G)$ is irreducible (it contains $\overline{A}(G)$), the representations r_q are never irreducible. In fact, their decomposition is now known to be of great importance for the theory of automorphic forms and group representations.

For the case when q is the norm form of a quadratic field, see [35], [45], [36], and [18]; for the case when q is the norm form of a division algebra in four variables see [37] and [18]. As already remarked in the Section 1, no such complete results have yet been obtained when q is a quadratic form in an odd number of variables. This is because until recently no one seriously attacked the representation theory of the two-fold covering groups of $SL_2(F)$.

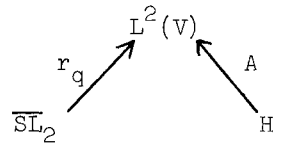
In these Notes we shall describe how a Weil representation in one and three variables decomposes and how its decomposition relates automorphic forms on $GL(1)$ and $GL(2)$ to automorphic forms on the metaplectic group. The general philosophy is explained in Subsection 2.4.

2.4. A philosophy for Weil's representation.

The purpose of this Subsection is to describe a simple principle which (though unproven) underlies most of the results of these Notes. Roughly speaking, the idea is that quadratic forms index 1-1 correspondences between automorphic forms on the metaplectic group and automorphic forms on the orthogonal group.

Let F denote a local or global field of characteristic zero, (q,V) a quadratic space over F, and r_q the corresponding representation of $SL_2(F)$. Let H denote the orthogonal group of q. This group acts on $L^2(V)$ through its natural action on V. The resulting representation of H in $L^2(V)$ may be assumed to be unitary and we denote it by $A(h)$.

Now fix F to be local. From Theorem 2.22 it follows that for $h \in H$ and $\overline{s} \in \overline{SL}_2(F)$ the operator $A(h)$ commutes with $r_q(\overline{s})$. In fact it seems plausible that r_q and A generate each others commuting algebras. Suppose for the moment that this is so. Then the primary constituents of r_q correspond 1-1 to primary constituents of A. In particular, the commuting diagram



leads to a correspondence between irreducible representations of H which occur in A and irreducible representations of $\overline{SL}_2$ which occur in r_q. Following R. Howe, we call this correspondence D for "duality". Globally, D should pair together automorphic forms on H which occur in A with automorphic forms on $\overline{SL}_2$ which occur in r_q. This is the correspondence alluded to above.

Since the existence of this correspondence D seems to rest on the hypothesis that r_q and A generate each others commuting algebras, some further remarks are in order. The precise

formulation of this hypothesis, in the greater generality of "dual reductive pairs", was communicated to me by Howe; as such, it is but one facet of his inspiring theory of "oscillator representations" for the metaplectic group (manuscript in preparation). In the context of SL_2, at least over the reals, this fact had also been suspected by S. Rallis and G. Schiffmann (cf. [52]). On the other hand, important special examples of D had already appeared in the literature. In [36] Shalika and Tanaka discovered that the Weil representation attached to the norm form of a quadratic extension of F yielded a correspondence between forms on SL(2) and SO(2). This seems to have been the first published example of D. (Actually the correspondence of Shalika-Tanaka was not 1-1 since they dealt with SO(2) instead of O(2).) For quadratic forms given by the norm forms of quaternion algebras over F the resulting corresponding between automorphic forms on the quaternion algebra and GL(2) was developed by Jacquet-Langlands ([18], Chapter III) following earlier work of Shimizu. In both cases, the fact that r_q and A generate each others commuting algebras was not established <u>apriori</u>; rather it appeared as a <u>consequence</u> of the complete decomposition of r_q.

One purpose of these Notes is to describe the duality correspondences belonging to two forms in an odd number of variables, namely $g_1(x) = x^2$, and $q_3(x_1,x_2,x_3) = x_1^2-x_1^2-x_3^2$. The inspiration for our discussion derives directly from general ideas of Howe's, an initial suggestion of Langland's, and earlier works of Kubota [ ], Shintani [ ], and Niwa [ ]. Our results, specifally Cor.4.18, Cor.4.20, and Theorem 6.3, lend further evidence to the general principle asserted above. However, as in earlier works, the fact that r_q and A generate each others commuting algebras, appears as a consequence of the existence of D.

2.5 Extending Weil's representation to GL_2

Weil's construction of the metaplectic group produces a multiplier representation r_q of SL_2. There are several reasons now why it is advantageous to extend this construction to GL_2. One is that Weil's representation for SL_2 often depends not only on q but also on the choice of additive character τ. By contrast, the natural analogue of r_q for GL_2 depends only on q.

In this Subsection I want to explain how Weil's original representation depends on τ. More precisely, I want to define an analogue of r_q for GL_2 which is independent of τ. The case when q is the norm form of a quadratic or quaternionic space is discussed in [18]. I shall treat the cases

$$q_1(x) = x^2$$

and

$$q_3(x_1,x_2,x_3) = x_1^2 - x_2^2 - x_3^2$$

following a suggestion of Cartier's.

Throughout this paragraph the following conventions will be in force:

1) F will be a local field of characteristic zero;

2) $n = 1$ or 3 according as $q = q_1$ or q_3;

3) r_n will denote r_{q_1} or r_{q_3};

4) $\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], 1\}$ will be abbreviated by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

Note that the formulas for r_n given in Theorem 2.22 depend on τ_t. Therefore, for emphasis, I shall denote r_n by $r_n(\tau_t)$; the notation r_n will be reserved for the representation eventually introduced for GL_2.

Proposition 2.27. (Cf. [18] Lemma 1.4.). If $a \in F^\times$, and $\overline{s} = \{s,\zeta\} \in \overline{SL}_2(F)$, define $\overline{s}^a$ to be $\{s^a, \zeta\, v(a,s)\}$ (cf. Proposition 2.6). Then

$$r_n(\tau_a)(\bar{s}) = r_n(\tau)(\bar{s}^a)$$

for all $a \in F^x$ and $\bar{s} \in SL_2(F)$.

Proof. We may assume without loss of generality that $n = 1$ and $\bar{s}$ is a generator of $\overline{SL}_2(F)$. Suppose first that $\bar{s} = \bar{w} = \{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], 1\}$. Then

$$\bar{s}^a = \bar{w}^a = \left[\begin{smallmatrix} 0 & a \\ -a^{-1} & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] \bar{w} \{1,(a,a)\}$$

and

$$r_n(\tau)(\bar{w}^a)\Phi(X) = (a,a)\, \gamma(q_n,\tau)(r_n(\tau)(\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right])\hat{\Phi})(X) .$$

Some tedious computations with Hilbert symbols also show that

$$\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] = \bar{w}\left[\begin{smallmatrix} 1 & a^{-1} \\ 0 & 1 \end{smallmatrix}\right] \bar{w} \left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right] \bar{w} \left[\begin{smallmatrix} 1 & a^{-1} \\ 0 & 1 \end{smallmatrix}\right]\{1,(a,a)\} .$$

Consequently

$$(2.43) \qquad r_n(\tau)(\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right])\Phi(X) = (a,a)\, |a|^{n/2} \frac{\gamma(q_n,\tau_a)}{\gamma(q_n,\tau)} \Phi(aX)$$

and

$$r_n(\tau)(\bar{w}^a)\Phi(X) = |a|^{n/2}\, \gamma(q_n,\tau_a)\hat{\Phi}(aX) .$$

But $|a|^{n/2}\gamma(q_n,\tau_a)\hat{\Phi}(aX) = r_n(\tau_a)(\bar{w})\Phi(X)$. Therefore $r_n(\tau)(\bar{w}^a) = r_n(\tau_a)(\bar{w})$, as was to be shown. The analogous identity for $\bar{s} = \{\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right], 1\}$ is completely straightforward. □

Corollary 2.28. The representation $r_n(\tau_a)$ is independent of $a \in F^x$ iff $r_n(\tau)$ extends to a representation of $\overline{GL}_2(F)$.

Proof. Suppose $r_n(\tau_a)$ is equivalent to $r_n(\tau)$ for all $a \in F^x$. Then for each $a \in F^x$ there is a unitary operator R_a in $L^2(F^n)$ such that

$$r_n(\tau_a)(\bar{s}) = R_a\, r_n(\tau) R_a^{-1}$$

for all $\bar{s} \in \overline{SL}_2(F)$. Equivalently, by Proposition 2.27,

$$r_n(\tau)(\overline{s}^a) = R_a\ r_n(\tau_a)R_a^{-1} \quad .$$

But $\overline{GL}_2(F)$ is the semi-direct product of $\overline{SL}_2(F)$ and $F^x=\{\left[\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}\right]\}$. Thus $r_n(\tau)$ can be extended to a representation of $\overline{GL}_2(F)$ by defining $r_n(\tau)(\left[\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}\right],1)$ to be R_a.

Conversely, if r_n extends to $\overline{GL}_2(F)$, $r_n(\tau)(\left[\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}\right],1)$ will intertwine $r_n(\tau)$ with $r_n(\tau_a)$. □

Example 2.29. If $a \in (F^x)^2$, say $a=\alpha^2$, then $R_a\Phi(x)=|\alpha|^{1/2}h(\alpha x)$ intertwines $r_1(\tau_a)$ and $r_1(\tau)$; in particular, $r_1(\tau)$ extends to a representation of

$$\overline{GL}_2{}^2(F) = \{\{g,\zeta\} \in \overline{GL}_2(F) : \det(g) \in (F^x)^2\} \quad .$$

In general, $r_q(\tau_a)$ will not be equivalent to $r_q(\tau)$. Indeed $r_q(\tau)$ will extend only to a representation of the subgroup

$$\{\{g,\zeta\} \in \overline{GL}_2(F) : \det(g) \text{ fixes } r_q(\tau)\} \quad .$$

To get around this problem we "fatten up" $r_q(\tau)$ before attempting to extend it. This way there is more room in the representation space for intertwining operators to act.

Definition 2.30. Let dt denote the restriction of additive Haar measure on F to F^x. Let r_q denote the direct integral of the representations $r_q(\tau_t)$ with respect to dt.

Note that the space of r_q is isomorphic to $L^2(F^n \times F^x)$. Moreover, the methods of Corollary 2.28 imply that r_q extends to a representation of $\overline{GL}_2(F)$ satisfying

$$r_q(\left[\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}\right])\Phi_t(X) = |a|^{-1/2}\Phi_{ta^{-1}}(X) \quad . \tag{2.44}$$

Proposition 2.31. The action of r_q in $L^2(F^n \times F^x)$ is given by the formulas

$$r_q(\left[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right])\Phi(X,t) = \psi_t(bq(X))\Phi(X,t) \tag{2.45}$$

(2.46)
$$r_q(\bar{w})\Phi(X,t) = \gamma(q,t)\hat{\Phi}(X,t)$$
$$= \gamma(q,t)\int_{F^n} \Phi(Y,t)\tau_t(q(X,Y)d_tY$$

(2.47)
$$r_q(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix})\Phi(X,t) = (a,a)|a|^{n/2}\frac{\gamma(q,\tau_{at})}{\gamma(q,\tau_t)}\Phi(aX,t)$$

and

(2.48)
$$r_q(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix})\Phi(X,t) = |a|^{-1/2}\Phi(X,ta^{-1}) .$$

Proof. Apply the definition of direct integral. □

Proposition 2.31 implies we could have defined r_q directly for GL_2 using formulas (2.45), (2.46), and (2.48). In fact this construction of r_q was communicated to me by Cartier awhile ago and my Definition 2.30 simply reformulates his ideas.

Note that

(2.49)
$$r_1(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix})\Phi(X,t) = \frac{\gamma(q_1,ta)}{\gamma(q_1,t)}|a|^{-1/2}\Phi(aX,ta^{-2}) .$$

Thus one easily checks that $r_1(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix})$ commutes with $r_1(\bar{g})$ for all $\bar{g} \in \overline{GL}_2(F)$ iff $\det(g) \in (F^X)^2$. This is consistent with Corollary 2.13(a).

Now it is natural to ask what shape the philosophy of Subsection 2.4 takes in the present context? Roughly speaking, the answer is that the orthogonal group of q is simply replaced by the group of similitudes of q.

Suppose first that $q = q_1$. The group of similitudes of q_1 is $F^X = GL_1(F)$ and its action in $L^2(F \times F^X)$ is given by

$$(A(y)\Phi)(x,t) = |a|^{-1/2}\Phi(ax,ta^{-2}) .$$

Note that $A(y)$ clearly commutes with r_1.

Now suppose $q = q_3$. If F^3 is realized as the space of 2×2 symmetric matrices with coefficients in F then $q(X) = \det(X)$.

The group of similitudes of q_3 is essentially $GL_2(F)$. More precisely, define

$$g \circ X = {}^t g\, X\, g$$

when $g \in GL_2(F)$ and $X \in F^3$. Then $q(g \circ X) = (\det g)^2 q(X)$. The action of $GL_2(F)$ in $L^2(F^3 \times F^\times)$ is given by

$$(A(g)\Phi)(X,t) = |\det g|^2 \Phi(g \circ X, t(\det g)^{-2})$$

and A once again commutes with r_3.

The result now is that r_1 (resp. r_3) should decompose into irreducible representations of $\tilde{G}$ indexed by irreducible representations of $GL_1(F)$ (resp. irreducible representations of $GL_2(F)$ which occur in A). Globally this means automorphic forms on the metaplectic group which occur in r_1 (resp. r_3) should be indexed by automorphic forms on GL_1 (resp. automorphic forms on GL_2 which occur in A). The global results that can be obtained or expected are described in Subsections 6.1 and 6.2. The local analysis is carried out in Subsections 4.3, 4.4, and 5.5.

2.6. Theta-functions

Classically, automorphic forms are constructed from theta-series attached to quadratic forms. This is the procedure reformulated in representation theoretic terms by Weil in [47]. For SL_2 the idea is this.

Suppose F is a number field and q is an F-rational quadratic form in n variables. Define a distributuon θ on F^n by

$$\theta(\Phi) = \sum_{\xi \in F^n} \Phi(\xi).$$

Piecing together local Weil representations a representation r_q of $\overline{SL}_2(\mathbb{A})$ is defined with the property that $\theta(r_q(\overline{g})\Phi)$ is $SL_2(F)$-invariant. In particular, $\theta(r_q(s)\Phi = \theta(\Phi)$ for all $s \in SL_2(F)$. This is the theta function attached to q. For GL_2, the point of departure is Proposition 2.32 below.

For v a non-archimedean place of F let O_v denote the ring of integers of F_v and U_v its group of units. Let $r_v(q)$ denote the corresponding Weil representation of $GL_2(F_v)$ in $L^2(F^n \times F^x)$. As in Subsection 2.1, K_v^N is the subgroup of $GL_2(O_v)$ consisting of matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $a \equiv 1$ and $c \equiv 0 \pmod N$. Here N is a positive integer divisible by 4. Thus $K_v^N = GL_2(O_v)$ if F_v has odd residual characteristic.

Proposition 2.32. Suppose $q = q_1$ or q_3. Then for almost every v, $r_v(q)$ is class 1, i.e., the restriction of $r_v(q)$ to $GL_2(O_v)$ has at least one fixed vector. More precisely, for each place v, define $\Phi_v^o \in L^2(F^n \times F^x)$ by

$$\Phi_v^o(x_1,\ldots,x_n,t) = \left(\prod_{i=1}^{n} 1_{O_v}\right) \otimes 1_{U_v^N}. \tag{2.50}$$

Here 1_{O_v} denotes the characteristic function of O_F and $1_{U_v^N}$ denotes the characteristic function of $U_v^n = \{y \in U_v : y \equiv 1 \pmod N\}$.

For all odd v,

$$r_v(q)(k)\Phi_o = \Phi_o \tag{2.51}$$

for $k \in K_v^N$.

Proof. The group $GL_2(O_v)$ is generated by the matrices

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad (b \in O_v)$$

$$w$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \quad (a \in U_v).$$

Thus it suffices to check (2.51) for these generators.

Recall that our canonical additive character τ_v has conductor O_v. In particular, $\tau_v(tbq(X)) = 1$ if $q(X) \in O_v$, $b \in O_v$, and $t \in U_v$. Therefore $r_v(q)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\Phi_v^o(X,t) = \Phi_v^o(X,t)$. Note also that the Fourier transform of 1_{O_v} is 1_{O_v}. Thus $r_v(q)(w)\Phi_v^o(X,t) = \Phi_v^o(X,t)$. The fact that $r_v(q)\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\Phi_v^o(X,t) = |a|^{-1/2}\Phi_v^o(X,ta^{-1}) = \Phi_v^o(X,t)$ is obvious since $a \in U_v$. □

To define r_q globally, and to introduce theta-functions on $\overline{G_{\mathbb{A}}}$, we need first to define an appropriate space of Schwartz-functions on $F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x$. We shall say that Φ on $F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x$ is "Schwartz-Bruhat" if $\Phi(X,t) = \prod_v \Phi_v(X_v,t_v)$ and:

(i) $\Phi_v(X_v,t_v)$ is infinitely differentiable and rapidly decreasing on $F_v^n \times F_v^x$ for each archimedean v;

(ii) for each finite v, Φ_v is the restriction to $F_v^n \times F_v^x$ of a locally constant compactly supported function on F_v^{n+1};

(iii) for almost every finite v, $\Phi_v = \Phi_v^o$ (the function defined by (2.50)).

Denote this space of functions by $\mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$. Since $\mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$ is dense in $L^2(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$, we can define a representation r_q of $\overline{G_{\mathbb{A}}}$ in $L^2(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^x)$ through the formula

$$r_q(\bar{g})\Phi = \otimes_v r_v(q)(\bar{g}_v)\Phi_v .$$

By virtue of Proposition 2.32 this definition is meaningful for at least all $\Phi \in \mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^\times)$. Indeed for almost all v, $\bar{g}_v \in GL_2(O_v)$ and $\Phi_v = \Phi_v^\circ$. Thus $r_v(q)(\bar{g}_v)\Phi_v = \Phi_v$ and $r_q(\bar{g})\Phi$ again belongs to $\mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^\times)$. By continuity, $r_q(\bar{g})$ operates (unitarily) in L^2. The role of τ_v in the local theory is now played by a non-trivial character τ of $F\backslash\mathbb{A}$.

The theta-functions on $\overline{G_{\mathbb{A}}}$ corresponding to q (or r_q) are defined as follows. For each $\Phi \in \mathcal{S}(F_{\mathbb{A}}^n \times F_{\mathbb{A}}^\times)$ define $\theta(\Phi,\bar{g})$ on $\overline{G_{\mathbb{A}}}$ by

$$\theta(\Phi,\bar{g}) = \sum_{(\xi,\eta)\in F^n\times F^\times} (r_q(\bar{g})\Phi)(\xi,\eta) .$$

Proposition 2.33. For each $\gamma \in GL_2(F)$,

$$\theta(\Phi,\gamma\bar{g}) = \theta(\Phi,\bar{g}); \tag{2.52}$$

i.e., $\theta(\Phi,\bar{g})$ is $GL_2(F)$ invariant.

Proof. We may assume without loss of generality that $\bar{g} = \{1,1\}$ and γ is a generator of $GL_2(F)$. Suppose first that $\gamma = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with $b \in F$. Then

$$r_q(\gamma)\Phi(X,t) = \tau(btq(X)\Phi(X,t)$$

with $\tau = \Pi\tau_v$ trivial on F. But q is F-rational. Thus $r_q(\gamma)\Phi(\xi,\eta) = \Phi(\xi,\eta)$ for $(\xi,\eta) \in F^n \times F^\times$ and (2.52) is immediate.

Now suppose $\gamma = w$. From Weil's theory (cf. Theorem 5 of [47]) it follows that $\gamma(q,\tau_t) \equiv 1$. Moreover, $|t| = 1$ if $t \in F^\times$. Therefore,

$$\Sigma\, r_q(w)\Phi(\xi,\eta) = \Sigma|\eta|^{n/2}\,\gamma(q,\tau_\eta)\hat{\Phi}(\eta\xi,\eta)$$

$$= \sum_{(\xi,\eta)} \hat{\Phi}(\eta\xi,\eta)$$

$$= \Sigma\ \hat{\Phi}(\xi,\eta)$$

$$= \Sigma\ \Phi(\xi,\eta).$$

The last step above follows from Poisson's summation formula.

Finally suppose $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ with $a \in F^{\times}$. By formula (2.48), $r_q(\gamma)\Phi(\xi,\eta) = |a|^{-1/2}\ \Phi(\xi,\eta a^{-1})$. But $|a|^{-1/2} = 1$. Therefore $\Sigma\ r_q(\gamma)\Phi(\xi,\eta) = \Sigma\ \Phi(\xi,\eta)$ as desired. □

Observation 2.34. The function $\theta(\Phi,\overline{g})$ always defines an automorphic form on $\overline{G_{\mathbb{A}}}$, i.e., $\theta(\Phi,\overline{g})$ is always G_F invariant. Nevertheless, it may often be the case that $\theta(\overline{g})$ is identically zero. Suppose, for example, that $\Phi(-X,t) = -\Phi(X,t)$, i.e. Φ is an odd function in X. Then $r_q(\overline{g})\Phi$ is also odd for all $\overline{g} \in \overline{G_{\mathbb{A}}}$. For simplicity, suppose also that $F = \mathbb{Q}$. From formula (2.47) it follows that $r_q(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})\Phi(X,t) = -\Phi(X,t)$, i.e.

$$\theta(\Phi,\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\overline{g}) = -\ \theta(\Phi,\overline{g}).$$

But by the G_F-invariance of θ,

$$\theta(\Phi,\overline{g}) = \theta(\Phi,\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\overline{g}).$$

Therefore $\theta(\Phi,\overline{g}) \equiv 0$.

We close this paragraph by demonstrating how $\theta(\Phi,\overline{g})$ generalizes the basic theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}.$$

Proposition 2.35. Fix $n = 1$, $F = \mathbb{Q}$, and $r_q = r_1$. Suppose $\Phi = \Pi\ \Phi_p$ where $\Phi_p = \Phi_p^{\circ}$ for all finite p and $\Phi_\infty(x,t) = |t|^{-1/4}\ e^{-|t|2\pi x^2}$. Then if $\overline{g} = \{\begin{bmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{bmatrix}, 1\}$,

$$\theta(\Phi,\overline{g}) = y^{1/4}\theta(x+iy).$$

Proof. Since $\begin{bmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix}$,

$$\sum_{(\xi,\eta)} r_q(\overline{g})\Phi(\xi,\eta) = \sum_{(\xi,\eta)} |y|^{1/4}\, e^{-2\pi y\xi^2}\, e^{2\pi i\xi^2 x\eta}.$$

Here we choose $\tau = \prod \tau_p$ with $\tau_\infty(x) = e^{2\pi ix}$ and τ_p trivial on O_p for $p < \infty$. Thus (ξ,η) contributes to the summation above only if $\xi \in \mathbb{Z}$ and $\eta = 1$ (recall $\Phi_2 = 1_{O_2} \cdot 1_{U_2^N}$). I.e.,

$$\theta(\Phi,\overline{g}) = y^{1/4} \sum_{n=-\infty}^{\infty} e^{2\pi in^2(x+iy)},$$

as was to be shown. □

Note that for $a \in (0,\infty)$, and Φ as above,

$$r(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix})\Phi(\xi,\eta) = |a|^{-1/2}\, \Phi(ax, ta^{-2})$$

$$= |a|^{-1/2}\, |ta^{-2}|^{-1/4}\, e^{-|t|2\pi x^2}$$

$$= |t|^{-1/4}\, e^{-|t|2\pi x^2} = \Phi(\xi,\eta).$$

Thus $\theta(\Phi,\overline{g})$ is actually defined on

$$\overline{X} = Z_\infty^0 \backslash G_F / \overline{G_A}.$$

§3. Automorphic Forms on the Metaplectic Group: Global Theory.

3.1. Preliminaries.

We start by describing the correspondence between classical forms of half-integral weight and automorphic forms on the metaplectic group of Section 2.2.

For convenience, we shall assume that the ground field is Q. Thus we shall consider functions $f(z)$, defined and holomorphic in $\{Im(z) > 0\}$, satisfying

$$f(\gamma z) = f(\frac{az+b}{cz+d}) = \chi(\gamma)(cz+d)^{k/2} f(z) \tag{3.1}$$

for all $\gamma \in \Gamma_1(N)$. Here, as before, k is an odd positive integer, N is a positive integer divisible by 4, and $\chi(\gamma)$ is the multiplier system of dimension $1/2$ described by (2.30).

We shall also assume that $f(z)$ is holomorphic at the cusps of $\Gamma_1(N)$. In fact we shall assume that $f(z)$ vanishes at these cusps, i.e., $f(z)$ is a cusp form. (For precise definitions, see [38].) Suffice it to say that cusp forms have Fourier expansions of the type

$$f(z) = \sum_{n=0}^{\infty} a(n)\exp(2\pi i\, n\, z) \tag{3.2}$$

with $a(0) = 0$, and for these forms,

$$|f(z)|\,Im(z)^{k/4} < M. \tag{3.3}$$

What we want to show is that such forms correspond to particularly nice functions on $\overline{G_{\mathbb{A}}}$. The space of these forms will be denoted by $S(k/2, \Gamma_1(N))$.

Recall that K_∞ denotes the maximal compact subgroup of $GL_2(\mathbb{R})$ whose connected component is

$$K_\infty^* = SO(2) = \{r(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\}$$

with $0 \leq \theta < 2\pi$. Although $\overline{G}_\infty$ does not split over $SO(2)$, $\overline{K^*_\infty}$ is still abelian. To describe its character group it is convenient to realize $\overline{K^*_\infty}$ as $\mathbb{Z}/4\pi\mathbb{Z}$ rather than as the group of pairs $\{\{r(\theta),\zeta\}\}$ with $r(\theta)$ as above. Thus

$$\overline{K^*_\infty} = \{\overline{r}(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\} \tag{3.4}$$

with $0 \leq \theta < 4\pi$. The isomorphism between these realizations of $\overline{K^*_\infty}$ must make $\{r(0),-1\}$ correspond to $\overline{r}(2\pi)$ since both these elements are of order two (and at most one non-trivial element of $\overline{K^*_\infty}$ can have this property). Consequently each character of $\overline{K^*_\infty}$ non trivial on Z_2 must be of the form

$$\overline{e}_{k/2}(\overline{r}(\theta)) = e^{i\frac{k}{2}\theta} \tag{3.5}$$

with k odd.

Next recall the Casimir operator for $\overline{SL_2}(\mathbb{R})$ in terms of the local parameterization

$$\overline{g} = (x,y,\theta).$$

Here $\overline{g} = \begin{bmatrix} y^{1/2} & x\,y^{-1/2} \\ 0 & y^{-1/2}\end{bmatrix} \overline{r}(\theta)$, $y > 0$, $x \in \mathbb{R}$, and $0 \leq \theta < 4\pi$.

The Casimir operator is the differential operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\,\frac{\partial^2}{\partial x \partial \theta} \quad . \tag{3.6}$$

Proposition 3.1. Each cusp form f in $S(k/2,\Gamma_1(N))$ corresponds to a function φ_f on $\overline{SL_2}(\mathbb{A})$ such that:

(1) $\varphi(\gamma\overline{g}) = \varphi(\overline{g})$ for $\gamma \in SL_2(\mathbb{Q})$ (recall $SL_2(\mathbb{Q})$ imbeds homomorphically in $SL_2(\mathbb{A})$ via the map $\gamma \to \{\gamma, S_{\mathbb{A}}(\gamma)\}$;

(2) $\varphi(\zeta\overline{g}) = \zeta\varphi(\overline{g})$ for $\zeta \in Z_2$, i.e., φ is a genuine function on $\overline{GL_2}(\mathbb{A})$;

(3) $\varphi(\overline{g}k_0) = \varphi(\overline{g})$ for all k_0 in $K_0^N \cap SL_2(\mathbb{A})$;

(4) $\varphi(\overline{g}\,\overline{r}(\theta)) = \overline{e}_{k/2}(\overline{r}(\theta))\varphi(\overline{g})$ for all $\overline{r}(\theta)$ in $\overline{K^*_\infty}$, i.e.,

φ transforms under $\overline{K_\infty^*}$ according to the character $\overline{e}_{k/2}$;

(5) viewed as a function of $\overline{SL}_2(\mathbb{R})$ alone, φ is smooth and satisfies the differential equation

$$\Delta\varphi = -\frac{k}{4}(\frac{k}{4} - 1)\varphi;$$

(6) φ is square integrable; more precisely,

$$\int_{SL_2(\mathbb{Q})\backslash\overline{SL}_2(\mathbb{A})/Z_2} |\varphi(\overline{g})|^2 \, d\overline{g} < \infty;$$

(7) φ is "cuspidal" on $\overline{SL}_2(\mathbb{A})$, i.e.

$$\int_{N_\mathbb{Q}\backslash N_\mathbb{A}} \varphi([\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}]\overline{g})dx = 0 \quad \text{(for a.e. } \overline{g}).$$

(This expression is meaningful since $N_\mathbb{A}$ lifts as a subgroup of $\overline{SL}_2(\mathbb{A})$.)

The significance of (6) and (7) is that φ_f will belong to the space of square-integrable cusp forms for the metaplectic group. The significance of (5) is that $-k/4(\frac{k}{4} - 1)$ will be the eigenvalue of the Casimir operator for the representation of $\overline{SL}_2(\mathbb{R})$ with lowest weight vector $k/2$ ($\overline{\pi}_{k/2}$ in the notation of Section 4).

To define φ_f and establish Proposition 3.1 we use a few Lemmas.

Lemma 3.2. Suppose $\overline{g} = (g, \zeta)$ belongs to $\overline{SL}_2(\mathbb{A})$. Then

$$\overline{g} = \gamma\overline{g}_\infty k_0$$

with $\gamma = \{\gamma, S_\mathbb{A}(\gamma)\}$ in $SL_2(\mathbb{Q})$, $k_0 = \{k_0, 1\}$ in $K_0^N \cap SL_2(\mathbb{A})$, and $\overline{g}_\infty$ in $\overline{SL}_2(\mathbb{R})$ determined up to left multiplication by $\{\gamma_0, S_\mathbb{A}(\gamma_0)\}$ in

$$\Gamma_1(N) = SL_2(\mathbb{Q}) \cap SL_2(\mathbb{R})K_0^N .$$

Proof. By "strong approximation" for SL(2), $SL_2(\mathbb{A}) = SL_2(\mathbb{Q})SL_2(\mathbb{R})K_0^N$. More precisely, $g = \gamma g_\infty k_0$, with g_∞ in $SL_2(\mathbb{R})$

determined up to left multiplication by elements of $\Gamma_1(N)$. Thus

$$\{g,\zeta\} = \{\gamma, S_{\mathbb{A}}(\gamma)\}\{g_\infty,\zeta_\infty\}\{k_0,1\}$$

with $\zeta_\infty = \zeta\,\beta_{\mathbb{A}}(\gamma,g_\infty)\beta_{\mathbb{A}}(\gamma g_\infty,k_0)S_{\mathbb{A}}(\gamma)$. If $\gamma_0=\{\gamma_0,S_{\mathbb{A}}(\gamma_0)\}\in\Gamma_1(N)$, straightforward but tedious computation shows that $\{g,\zeta\} = \{\gamma\gamma_0^{-1}, S_{\mathbb{A}}(\gamma\gamma_0^{-1})\}\{\gamma_0 g_\infty,\zeta'_\infty\}\{k_0,1\}$, with $\{\gamma_0 g_\infty,\zeta'_\infty\} = \{\gamma_0,S_{\mathbb{A}}(\gamma_0)\}\{g_\infty,\zeta_\infty\}$. □

For $g = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ in $SL_2(\mathbb{R})$, recall that $J^*(g,z) = (cz+d)^{1/2}$ does not define a factor of automorphy for $\Gamma_1(N)$. (We agreed to choose $w^{1/2}$ so that $-\pi/2 < \arg(w^{1/2}) \leq \pi/2$.) Indeed by the functional equation for the theta-function (Hecke [16], pp. 919-940),

$$\frac{\theta(\gamma z)}{\theta(z)} = \chi(\gamma)(cz+d)^{1/2}$$

for γ in $\Gamma_1(N)$.

Now we can define φ_f. For $\overline{g}_\infty = (g_\infty,\zeta)$ in $\overline{SL}_2(\mathbb{R})$, set

$$J^*(\overline{g}_\infty,z) = \zeta J^*(g,z),$$

and for $f(z)$ in $S(k/2,\Gamma_1(N))$, set

$$\varphi_f(\overline{g}) = f(\overline{g}_\infty(i))J^*(\overline{g}_\infty,i)^{-k}. \tag{3.7}$$

Note that in (3.7),

$$\overline{g}_\infty(z) = \frac{az+b}{cz+d}$$

if $\overline{g}_\infty = (\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right],\zeta)$. Thus (3.1) rewrites itself as

$$f(\overline{\gamma}(z)) = J^*(\overline{\gamma},z)^k f(z) \tag{3.8}$$

for all $\overline{\gamma} = \{\gamma,S_{\mathbb{A}}(\gamma)\}$ in $\Gamma_1(N)$. (By Proposition 2.16, $S_{\mathbb{A}}(\gamma) = \chi(\gamma)$!)

Lemma 3.3. (a) $J^*(\overline{g}_\infty,z)$ defines a factor of automorphy on $\overline{SL}_2(\mathbb{R})$, i.e.

(3.9) $$J^*(\overline{g}\ \overline{g}',z) = J^*(\overline{g},\overline{g}'(z))J^*(\overline{g}',z)$$

for all $\overline{g},\overline{g}'$ in $SL_2(\mathbb{R})$;

(b) The restriction of $J^*(g,i)^k$ to $\overline{K}_\infty^*$ defines a character of $\overline{K}_\infty^*$ non-trivial on Z_2, namely $\overline{e}_{k/2}$.

Proof. (a) The function

$$F(g,g') = \frac{J^*(g\ g',z)}{J^*(g,g'(z))J(g',z)}$$

takes its values in Z_2 and is obviously a Borel map on $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$. Since F can also be shown to satisfy the factor set relations (2.1) and (2.2) (cf. Maass [24], pp. 115-116), J^* defines a factor of automorphy on the extension of $SL_2(\mathbb{R})$ determined by F. Further computations then show that F is the cocycle α described in Theorem 2.2.

(b) By (a), and the fact that $\overline{r}(\theta)$ stabilizes $i \in \{Im(z) > 0\}$, the restriction of $(J^*)^k$ to $\overline{K}_\infty^*$ clearly defines a character of $\overline{K}_\infty^*$. But by definition, $J^*((1,\zeta),i)^k = \zeta^k = \zeta$. That $J^* = \overline{e}_{1/2}$ follows from the definition of $\overline{e}_{1/2}$. □

Proof of Proposition 3.1.

Note first that $\varphi_f(\overline{g})$ is well defined even though $\overline{g_\infty}$ in (3.7) is determined only modulo $\Gamma_1(N)$. Indeed by (3.8) and (3.9),

$$f(\gamma_0\overline{g_\infty}(i))J^*(\gamma_0\overline{g_\infty},i)^{-k} =$$

$$J^*(\gamma_0,\overline{g_\infty}(i))^k f(\overline{g_\infty}(i))J^*(\gamma_0,\overline{g_\infty}(i))^{-h}J^*(\overline{g_\infty},k)^{-k}$$

$$= \varphi_f(\overline{g})$$

for each γ_0 in $\Gamma_1(N)$.

Properties (1) - (4) are trivial. To prove (5), recall our analyticity assumption for $f(z)$. Then use straightforward computation (applied to $\varphi_f(x,y,\theta) = y^{k/4}f(x+iy)e^{-i(k/2)\theta}$) to verify that

$$\Delta\varphi_f(\overline{g}) = \frac{ik}{2} e^{-\frac{ik}{2}\theta} y^{\frac{k}{4}+1} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z)$$

$$- \frac{k}{4}\left(\frac{k}{4} - 1\right) y^{k/4} f(z) e^{-\frac{ik}{2}\theta} .$$

To prove (6), note by (1)-(4) that $|\varphi_f(g)|^2$ actually defines a function on the homogeneous space

$$SL_2(\mathbb{Q}) \backslash \overline{SL}_2(\mathbb{A}) / K_0^N \overline{K_\infty^*}$$

which by Lemma 3.2 is isomorphic to

$$\Gamma_1(N) \backslash \overline{SL}_2(\mathbb{R}) / \overline{K_\infty^*} \quad .$$

Consequently

$$\int |\varphi_f(g)|^2 dg = (\text{constant}) \iint |f(z)|^2 y^{k/2} \frac{dxdy}{y^2} \tag{3.10}$$

and the right-hand side of (3.10) is finite by (3.3).

To prove (7), simply compute:

$$\begin{aligned} \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\overline{g}\right) dx &= \int_{\mathbb{Z}\backslash\mathbb{R}} \varphi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\overline{g}\right) dx \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\overline{g_\infty}(i)\right) J^*\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\overline{g_\infty}, i\right)^{-k} dx \\ &= J^*(\overline{g_\infty}, i)^{-k} a(0) \end{aligned}$$

where $a(0)$ is the zeroth Fourier coefficient of f. This completes the proof of Proposition 3.1. □

Remark. The Petersson inner product in $S(k/2;\Gamma_1(N))$ is given by the integral

$$\iint_F f(z)\overline{g(z)} y^{k/2} \frac{dxdy}{y^2}$$

with F a fundamental domain for $\Gamma_1(N)$ in H. This integral diverges in general for non-cuspidal modular forms in $M(k/2,\Gamma_1(N))$. It converges, however, for modular forms of weight $1/2$ $(k = 1)$.

We conclude this paragraph by remarking that Proposition 3.1, with the obvious modifications, is valid for $\overline{G_{\mathbb{A}}}$ in place of $\overline{SL_2}(\mathbb{A})$. For $\overline{G_{\mathbb{A}}}$, one must appeal to the fact that

$$G_{\mathbb{A}} = G_{\mathbb{Q}} \, G_{\infty}^{0} \, K_{0}^{N} \tag{3.11}$$

where G_{∞}^{0} denotes the connected component of the identity in $GL_2(\mathbb{R})$. The result is that functions on $\overline{G_{\mathbb{A}}}$ which are trivial on Z_{∞}^{0} and satisfy (1)-(7) of Proposition 3.1 again correspond to classical forms of weight $k/2$.

Proposition 3.1 also generalizes to arbitrary number fields. However, (3.11) must be replaced by the identity

$$G_{\mathbb{A}} = \bigcup_{j=1}^{h} G_{\mathbb{Q}} a_j G_{\infty}^{0} K_{0}^{N}$$

with $a_j \in G_{\mathbb{A}}$. Consequently one starts with h classical forms f_j of weight $k/2$ and defines φ on $\overline{G_{\mathbb{A}}}$ by applying (3.7) to f_j and each piece $G_{\mathbb{Q}} a_i G_{\infty}^{0} K_{0}^{N}$. The result is that functions on $\overline{G_{\mathbb{A}}}$ trivial on Z_{∞}^{0} and satisfying (1)-(7) of Proposition 3.1 now correspond to <u>collections</u> of classical forms of weight $k/2$. The only "new" feature of this general situation is already apparent in the case of totally imaginary fields. In this case f is defined on a product of "quaternionic half-spaces" and <u>the corresponding metaplectic functions</u> φ can be K_{∞} invariant. For details we refer the reader to Proposition 5 of Kubota [21].

3.2 Factorization of Automorphic Forms.

Let $\overline{X}$ denote the homogeneous space $Z_\infty^0 G_F \backslash \overline{G}_A$, as in Section 2.2. According to Proposition 3.1, forms in $S(k/2, \Gamma_1(N))$ correspond to special functions in $\overline{L}^2(\overline{X})$, the space of square-integrable functions on $\overline{X}$ satisfying $\varphi(\zeta\overline{g}) = \zeta\varphi(\overline{g})$ for $\zeta \in Z_2$. Inspired by Jacquet-Langlands' theory for $GL(2)$, we now make the following definition.

Definition 3. An automorphic representation of the metaplectic group (or a generalized automorphic form of half-integral weight) is an irreducible unitary representation of $\overline{G}_A$ which occurs in the decomposition of the regular representation $\overline{T}$ of $\overline{G}_A$ in $\overline{L}^2(\overline{X})$.

Our purpose in this paragraph is to explain how such representations (like automorphic forms for $GL(2)$) factor as "products" of representations of the local groups $\overline{G}_v$. Thus we shall first recall some basic facts from representation theory tailor fit to the groups $\overline{G}_v$.

Suppose $\overline{\pi}_v$ is an irreducible unitary "genuine" representation of $\overline{G}_v$. Suppose also that v is finite and odd. Then we say that $\overline{\pi}_v$ is class 1 if its restriction to $\overline{K}_v$ contains the representation

$$\overline{\sigma}_0 : (k,\zeta) \to \zeta$$

at least once. Equivalently, the restriction of this representation to K_v contains the trivial representation of K_v at least once. As we shall see in Section 6, such representations will contain the identity representation exactly once.

To make sense out of a product of the form

$$\bigotimes_v \pi_v,$$

where the product is over all places of F, and $\overline{\pi}_v$ is irreducible, unitary, and genuine for all v, we must suppose that $\overline{\pi}_v$ is class 1 for almost all v, say for $v \notin S_0$.

Let $\{\pi'_v\}$ denote the collection of β_v-representation of G_v determined by $\{\overline{\pi}_v\}$ above through Proposition 2.18 and let H_v denote the space of $\overline{\pi}_v$ (or π'_v). For each $v \notin S_0$, pick a unit vector ξ_v^0 in H_v fixed by K_v. By the above remarks, ξ_v^0 is uniquely determined up to a scalar of modulus 1. For each finite set $S \supset S_0$, set H_S equal to $\underset{v\in S}{\otimes} H_v$. Then if $S' \supset S$, there is a natural isometric imbedding of H_S in $H_{S'}$, namely $\xi \to \xi \otimes (\underset{v\in S'-S}{\otimes} \xi_v^0)$. Consequently we can define the pre-Hilbert space $\varinjlim_S H_S$, and (by completion) a Hilbert space $H = \hat{\otimes} H_v$ on which the β_A-representation

$$\pi' = \underset{v}{\otimes} \pi'_v$$

acts as follows: If $\xi = \underset{v}{\otimes} \xi_v$ is any decomposable element of H, with $\xi_v = \xi_v^0$ for almost every v, then

$$\pi'(g)\xi = \otimes \pi'_v(g_v)\xi_v$$

for $g = (g_v)$ in $G_{\mathbb{A}}$.

By the construction above we obtain from $\{\overline{\pi}_v\}$ a unitary $\beta_{\mathbb{A}}$-representation of $G_{\mathbb{A}}$ which can easily be shown to be irreducible (by Schur's Lemma). The unitary "genuine" representation of $\overline{G}_{\mathbb{A}}$ determined by it is also irreducible. We shall denote it by

$$\overline{\pi} = \underset{v}{\otimes} \overline{\pi}_v$$

and refer to $\overline{\pi}$ as the "tensor product of the representations $\overline{\pi}_v$" (even though this product makes sense only when viewed as a product of corresponding multiplier representations of G_v). Similarly, given an irreducible unitary representation $\overline{\pi}$ of $\overline{G}_{\mathbb{A}}$ we say that $\overline{\pi}$ is <u>factorizable</u> if it can be realized as $\otimes \overline{\pi}_v$ in this sense.

<u>Theorem 3.5</u>. Every irreducible unitary genuine representation of $\overline{G}_{\mathbb{A}}$ is factorizable. I.e., there exists a family of irreducible unitary representations $\overline{\pi}_v$ of $\overline{G}_v$, completely determined by $\overline{\pi}$,

genuine for every v, and class 1 for almost every v, such that

$$\overline{\pi} = \otimes \overline{\pi}_v$$

in the above sense.

To prove this Theorem we need to appeal to the fact that each local group $\overline{G}_v$ is type 1. That is, each factor representation of $\overline{G}_v$ is primary. For a proof of this fact for representations of $\overline{G}_v$ which are trivial on Z_2 see Howe [17] and Harish-Chandra [15]. The Lemma which we shall use to exploit it is:

Lemma 3.6. Suppose G is a product of locally compact groups G_1 and G_2. Suppose also that the cocycle β on G is a product of cocycles β_1 and β_2 on G_1 and G_2, and that every β_i-representation of G_i which is a factor is type I (for $i = 1,2$). Then: every irreducible (unitary) β-representation of G is of the form $\pi_1 \otimes \pi_2$ with π_i an irreducible unitary β_i-representation of G_i completely determined (up to isomorphism) by π.

This Lemma is implicit in Mackey [24]. Using it we can sketch the proof of Theorem 3.5 following the spirit of [9] and [11].

First, fix a place v and write $G_{\mathbb{A}}$ as the direct product $G_{v_0} \times G'_{v_0}$, where G'_v consists of adeles (g_{v_0}) with $g_{v_0} = 1$. By Lemma 3.6, π', the $\beta_{\mathbb{A}}$-representation of $G_{\mathbb{A}}$ determined by $\overline{\pi}$, is the tensor product of an irreducible unitary β_{v_0}-representation of G_{v_0} and an irreducible β'_{v_0}-representation of G'_{v_0}. Thus we obtain a collection of irreducible unitary β_v-representations of G_v for each place v.

Next we need to show that these π'_v contain a vector invariant for K_v for almost every v. For simplicity, assume that $F = \mathbb{Q}$ so v runs through the sequence of rational primes. Let K_v^0 denote the subgroup of adeles of $G_{\mathbb{A}}$ of the form $(1,\ldots,1,k_v,\ldots,k_w,\ldots)$ where $k_w \in K_w$ for $w \geq v$. Clearly K_v^0 converges to the identity

as $v \to \infty$. Thus one can show that H (the space of π') contains a vector invariant for K_v^0 when v is sufficiently large, say greater than or equal to v_0. From this it follows that π_v' is class 1 for each $v \geq v_0$.

Finally we need to check that $\pi' = \otimes \pi_v'$. So choose an orthonormal basis $\{f_{v,j}\}$ in each H_v such that $f_{v,1}$ is invariant for K_v for each $v > v_0$. Since π' and $\otimes \pi_v'$ are both irreducible it will suffice to construct an isometry from $\otimes H_v$ to H compatible with the actions of $G_{\mathbb{A}}$.

By Lemma 3.6, H can be expressed as $(\underset{w<v}{\otimes} H_w) \otimes H_v'$ <u>for each v</u>. Thus for each $v \geq v_0$, some vector in H invariant for K_v is of the form $(\underset{w<v}{\pi} v_{w,1}) f_v'$ with f_v' a unit vector in H_v' invariant for K_v. On the other hand, $\underset{w<v}{\otimes} H_w = H_v \otimes (\underset{w<v}{\otimes} H_v)$, and $f_{v,1}$ is (up to a scalar) the <u>only</u> vector in H_v invariant under K_v. Thus for each $v \geq v_0$ we get an imbedding of $(\underset{w<v}{\otimes} H_w)$ into H (namely $f \to f(\underset{w>v}{\pi} f_{w,1})$) with the required compatibility conditions satisfied. □

3.3 The Spectrum of the Metaplectic Group

Let $L^2(\overline{X})$ denote the Hilbert space of measurable functions on $G_F \backslash G_{\mathbb{A}}$ which are trivial on Z_∞^0 and square-integrable on

$$\overline{X} = Z_\infty^0 G_F \backslash \overline{G}_A,$$

a quotient space of finite volume. The measure on $\overline{X}$ is such that

$$\int_{\overline{X}} f(\overline{g}) d\overline{g} = \int_X \{ \sum_{\zeta \in Z_2} f((g,\zeta)) \} dg$$

if $X = \overline{X}/Z_2 = G_F \backslash GL_2(\mathbb{A})/Z_\infty^0$

Let $L^2(X)$ denote the subspace of Z_2-invariant functions in $L^2(\overline{X})$ and $\overline{L}^2(\overline{X})$ its orthocomplement. Basic facts concerning the decomposition of $L^2(X)$ are well-known but we recall them here for future reference. For details, see [18] and [17].

Let $B'_{\mathbb{A}}$ denote the non-unimodular subgroup $N_{\mathbb{A}} A_F A_\infty^0$ of $B_{\mathbb{A}}$. (A_∞^0 is the group of positive diagonal matrices in A_∞.) For each complex number s, let $e^{sH(b)}$ denote the character

$$b = \begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} \to \left| \frac{a_1}{a_2} \right|^{s/2}$$

of $B'_{\mathbb{A}}$. Here $H(b) = \log \left| \frac{a_1}{a_2} \right|^{1/2}$. In general, suppose

$$g = n h_t \, a k z \tag{3.12}$$

with $n \in N_{\mathbb{A}}$, $h_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$, $a \in A^1_{\mathbb{A}} = \{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in A_{\mathbb{A}} : |a_1| = |a_2| = 1 \}$, $k \in K = K_\infty^* \prod_{v < \infty} K_v$, and $z \in Z_\infty^0$. Then $H(g) = t$.

Let $H(s)$ denote the space of complex-valued measurable functions φ on $G_{\mathbb{A}}$ such that

$$\int_K |\varphi(k)|^2 \, dk < \infty \tag{3.13}$$

and

$$\varphi(bg) = e^{(s+1)H(b)}\varphi(g) = \left|\frac{a_1}{a_2}\right|^{\frac{s+1}{2}} \varphi(g) \tag{3.14}$$

for all $b \in B'_{\mathbb{A}}$. The induced representation

$$R(x,s) = \mathrm{Ind}(B'_{\mathbb{A}}, G_{\mathbb{A}}, e^{sH(b)})$$

operates in this space by right translation and is unitary when $\mathrm{Re}(s) = 0$. Moreover $R(x,s)$ and $R(x,-s)$ are equivalent. The basic fact is that the direct integral

$$\int_{\substack{\mathrm{Re}(s)=0 \\ \mathrm{Im}(z)>0}} R(x,s)d|s| \tag{3.15}$$

intertwines with a subrepresentation T_c of the regular representation T of $G_{\mathbb{A}}$ in $L^2(X)$ but to prove this one needs to introduce Eisenstein series for each φ in $\mathbb{H}(s)$.

More precisely, if $\phi \in C^\infty(B'_{\mathbb{A}}\backslash G_{\mathbb{A}})$, ϕ is said to be <u>of type τ</u> if for each $x \in G_{\mathbb{A}}$, $\phi(xk)$ is contained in a subspace of $L^2(K)$ on which the regular representation of K is equivalent to τ. The space of such ϕ will be denoted (τ) and the collection of all such (τ) by L. Thus $\bigoplus_L (\tau) = \mathbb{H}$, the Hilbert space of measurable ϕ on $B'_{\mathbb{A}}\backslash G_{\mathbb{A}}$ satisfying (3.13).

If $\phi \in (\tau) \subset \mathbb{H}$, then $\phi(g)e^{(s+1)H(g)}$ belongs to the space of $R(x,s)$ <u>for all $s \in \mathbb{C}$</u>, and the <u>Eisenstein series associated to ϕ</u> is by definition

$$E(g,\phi,s) = \sum_{\gamma \in B_F\backslash G_F} \phi(\gamma g)e^{(s+1)H(\gamma g)}.$$

However, this series converges only if $\mathrm{Re}(s) > 0$. Thus $E(g,\phi,s)$ is defined for $\mathrm{Re}(s) = 0$ only in the sense of analytic continuation.

It is known that $E(g,\phi,s)$ is meromorphic in the whole s-plane and satisfies the functional equation

$$E(g,\emptyset,s) = E(g,M(s)\emptyset,-s)$$

where $M(s)$ is the "τ-component" of the operator intertwining $R(x,s)$ with $R(x,-s)$. Moreover, the properties of $M(s)$ imply that $E(g,\emptyset,s)$ is regular in $\mathrm{Re}(s) \geq 0$ except possibly at $s = 1$. In particular, the function $E(g,\emptyset,s)$ is well-behaved <u>on X</u> when $\mathrm{Re}(s) = 0$ and may be used to intertwine (3.15) with T_c. The orthocomplement of the space of T_c then has <u>discrete</u> spectrum, is denoted $L^2_d(X)$, and may be described as follows.

Let $L^2_0(X)$ denote the closed subspace of functions in $L^2(X)$ satisfying the cuspidal condition

$$\int_{F\backslash A} \varphi(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g)dx = 0$$

for a.e. $g \in G_A$. This is the space of square integrable cusp forms for G_A. If $L^2_0(E)$ denotes the subspace of functions in $L^2(X)$ which are realizable as residues of $E(g,\emptyset,s)$ at $s = 1$, then

$$L^2_d(X) = L^2_0(X) \oplus L^2_0(E),$$

and this decomposition is compatible with G_A.

Before returning to the metaplectic group note that $E(g,\emptyset,s)$ has a pole at $s = 1$ <u>only if</u> the τ-type of $\emptyset$ is $\tau(k) = \chi(\det k)$ (with χ a character of $F^x\backslash A^x$ whose square is trivial on F^0_∞). In particular, suppose μ_1 and μ_2 are quasi-characters of F^x_v such that $\mu_1\mu_2^{-1}(x) = |x|$ and let $\rho(\mu_1,\mu_2)$ denote the representation of G_v induced from the character

$$\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} \to \mu_1(a_1)\mu_2(a_2)$$

of B_v. Then each constituent of $L^2_0(E)$ is of the form $\otimes \pi_v$ with π_v equal to the one-dimensional quotient of some such $\rho(\mu_1,\mu_2)$.

Now we return to $\overline{G}_{\mathbb{A}}$ and the natural representation $\overline{\Pi}$ of $\overline{G}_{\mathbb{A}}$ in $\overline{L}^2(\overline{X})$. This latter space consists of functions φ in $L^2(\overline{X})$ which are genuine, i.e. satisfy $\varphi(\zeta\overline{g}) = \zeta\varphi(\overline{g})$ for $\zeta \in Z_2$. The crucial point is that $\overline{G}_{\mathbb{A}}$ splits over $B_{\mathbb{A}}'$. Thus one can introduce Eisenstein series and induced representations as for $G_{\mathbb{A}}$.

To be more specific, for arbitrary $\overline{g} \in \overline{G}_{\mathbb{A}}$, write

$$\overline{g} = nh_t\overline{a}\overline{k}z$$

where $\overline{a} \in \overline{A}_{\mathbb{A}}^1$, $\overline{k} \in \overline{K}$, and n, h_t, and z are as in (3.12). Then the function $H(\overline{g}) = t$ is again well-defined. In particular,

$$H(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix},\zeta) = \log\left|\frac{a_1}{a_2}\right|^{1/2}$$

and $\delta(\overline{b}) = e^{2H(\overline{b})}$ describes the modular function for $\overline{B}_{\mathbb{A}}$.

Let $\overline{\mathcal{H}}$ denote the Hilbert space of measurable functions ϕ on $B_{\mathbb{A}}'\backslash\overline{G}_{\mathbb{A}}$ such that $\varphi(\overline{g}\zeta) = \zeta\varphi(\overline{g})$ for $\zeta \in Z_2$, and

$$\int_K |\varphi(k)|^2 dk < \infty.$$

(By Iwasawa, $\overline{G}_{\mathbb{A}} = \overline{B}_{\mathbb{A}}\overline{K}$). If $\overline{\tau}$ is an irreducible unitary representation of $\overline{K}$ define $\overline{\tau}$ and $\overline{L}$ as before so that $\overline{\mathcal{H}} = \bigoplus_{\tau} (\overline{\tau})$.

Now fix $\phi \in \overline{\mathcal{H}}$ and set

$$E(\overline{g},\phi,s) = \sum_{\gamma\in B_F\backslash G_F} \phi(\gamma\overline{g})e^{(s+1)H(\gamma\overline{g})}. \tag{3.16}$$

This is an Eisenstein series on the metaplectic group. It makes sense since $\overline{G}_{\mathbb{A}}$ splits over G_F and it converges for $\mathrm{Re}(s) > 1$ to a function on $Z_\infty^0\backslash\overline{G}_{\mathbb{A}}$ which is left invariant for G_F. As such, $E(\overline{g},\phi,s)$ generalizes classical series of the type

$$E(z,k,s') = \sum_{c,d} \left(\frac{c}{d}\right)\left(\frac{cz+d}{|cz+d|}\right)^{k/2}|cz+d|^{-2s'} \tag{3.17}$$

and

$$E(u,0,s) = \sum_{\Gamma_\infty \backslash \Gamma(N)} \chi(\gamma)\, v(\gamma u)^{s+1}. \tag{3.18}$$

This last function is an Eisenstein series over $\mathbb{Q}(\sqrt{-d})$ which in our set-up corresponds to some $E(\overline{g},\emptyset,s)$ with $\emptyset$ of "trivial" $\overline{K}$-type. Such series were studied in [21] and recalled in Section 1.

The function described by (3.17) corresponds to an Eisenstein series for the metaplectic group defined over $\mathbb{Q}$. This series was investigated first by Siegel in [42], then more recently by Shimura in [39].

In general, these Eisenstein series intertwine certain induced representations with a subrepresentation of $\overline{T}$. To be more precise, let $\zeta e^{sH(\overline{b})}$ denote the character

$$\overline{b} = \left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}, \zeta\right) \to \zeta e^{sH(\overline{b})}$$

by $\overline{B}'_{\mathbb{A}}$. Then the induced representation

$$\overline{R}(\overline{x},s) = \mathrm{Ind}(\overline{B}'_{\mathbb{A}},\ \overline{G}_{\mathbb{A}},\ \zeta e^{sH(\overline{b})})$$

is unitary when $\mathrm{Re}(s) = 0$ and satisfies

$$R(\overline{x}\zeta, s) = \zeta R(\overline{x}, s)$$

for all $s \in \mathbb{C}$. ($R(\overline{x},s)$ is a genuine representation of $\overline{G}_{\mathbb{A}}$.) Its space $\overline{\mathcal{H}}(s)$ consists of functions φ on $\overline{G}_{\mathbb{A}}$ such that

$$\varphi(\overline{b}\ \overline{g}) = \zeta e^{(s+1)H(\overline{b})}\varphi(\overline{g})$$

for all $b \in \overline{B}'_{\mathbb{A}}$, and

$$\int_{\overline{K}} |\varphi(k)|^2\, dk < \infty.$$

Note that if $\emptyset \in \overline{\mathcal{H}}$, the function $\emptyset(\overline{g})e^{(s+1)H(\overline{g})}$ belongs to

the space of $R(\overline{x},s)$ In fact the correspondence

$$I(s): \phi(\overline{g}) \to \phi(\overline{g})e^{(s+1)H(\overline{g})}$$

is an isomorphism between $\overline{\mathbb{H}}$ and $\overline{\mathbb{H}}(s)$ and it is easy to check that

(3.19) $$\overline{T}(\overline{x})E(\overline{g},\phi,s) = E(\overline{g},I^{-1}(s)\ \overline{R}(\overline{x},s)\ I(s)\phi,s).$$

Clearly (3.19) says that E intertwines $\overline{R}(\overline{x},s)$ with a subrepresentation of $\overline{T}$. However, as for $GL(2)$, this statement is meaningless unless the analytic continuation of $E(\overline{g},\phi,s)$ is exploited.

Roughly speaking, the analytic continuation and functional equation of $E(\overline{g},\phi,s)$ makes it possible to intertwine

$$\int_{\substack{\mathrm{Re}(s)=0\\ \mathrm{Im}(s)>0}} \overline{\mathbb{H}(s)}\ d|s|$$

with an invariant subspace of $\overline{L}^2(\overline{X})$ whose orthocomplement decomposes discretely. This orthocomplement, call it $\overline{L}^2_d(\overline{X})$, will be the direct sum of $\overline{L}^2_0(X)$, the space of functions in $\overline{L}^2(\overline{X})$ satisfying the cuspidal condition (7) of Proposition 3.1, and a space $\overline{L}^2_0(E)$ consisting of residues of $E(\overline{x},\phi,s)$ at poles at the right of $\mathrm{Re}(s) = 0$.

These poles of $E(\overline{x},\phi s)$ coincide with those of the operator $\overline{M}(s)$ intertwining $\overline{R}(\overline{x},s)$ with $\overline{R}(\overline{x},-s)$. In [10] we show they occur only at $s = 1/2$ and generate representations which are simple to describe. A new feature of the decomposition of $\overline{L}^2(\overline{X})$ (as opposed to $L^2(X)$) is that in the latter only infinite-dimensional (as opposed to one-dimensional) representations occur outside the space of cusp forms. Thus if $\overline{\pi} = \otimes\ \overline{\pi}_v$ so occurs it is natural to ask what its local components are? This last question really just concerns Eisenstein series. The remarks below are included to explain the question and to show why Sections 4 and 5 are prerequisites for a meaningful response to it.

Note added (March 1976). Precise results concerning the analytic continuation and poles of $E(\overline{x},\emptyset,s)$ were announced in [10]. Complete proofs of these results (at least for certain special choices of $\emptyset$) will appear as part of a joint work with Jacquet [54].

3.4. Odds and Ends.

Recall that $\overline{R}(\overline{x},s)$ is the representation of $\overline{G}_{\mathbb{A}}$ induced from the character $\zeta e^{sH(\overline{b})}$ of $\overline{B}'_{\mathbb{A}} = N_{\mathbb{A}} A_F A^0_\infty Z_2$. Thus by inducing in stages,

$$R(\overline{x},s) = \mathrm{Ind}(\overline{B}_{\mathbb{A}}, \overline{G}_{\mathbb{A}}, T(\overline{b},s))$$

Here $T(\overline{b},s)$ denotes the representation of $\overline{B}_{\mathbb{A}}$ (the full triangular subgroup of $\overline{G}_{\mathbb{A}}$) induced from $\zeta e^{sH(\overline{b})}$.

Now let $R(\overline{A})$ denote the subrepresentation of the natural representation of $\overline{A}^1_{\mathbb{A}}$ in $L^2(A_F A^0_\infty \backslash \overline{A}_{\mathbb{A}})$ which acts on genuine functions on $\overline{A}_{\mathbb{A}}$. Since the quotient $A_F A^0_\infty \backslash \overline{A}_{\mathbb{A}}$ is compact,

$$R(\overline{A}) = \bigoplus_{\overline{\lambda}} m_{\overline{\lambda}} \overline{\lambda}$$

with the summation extending over the "genuine" dual group of $\overline{A}^1_{\mathbb{A}}$ ($m_{\overline{\lambda}}$ is a non-zero integer for only countably many $\overline{\lambda}$).

If we view $T(\overline{b},s)$ as a representation of $\overline{A}_{\mathbb{A}}$ it is easy to see that $T(\overline{b},s)$ is the tensor product of $R(\overline{A})$ with the one dimensional representation $\zeta e^{sH(a)}$ of A^0_∞. Thus

$$R(\overline{x},s) = \bigoplus_{\overline{\lambda}} R(\overline{x},\overline{\lambda},s). \tag{3.20}$$

In (3.20), $\overline{\lambda}$ denotes an irreducible unitary representation of $\overline{A}^1_{\mathbb{A}}$ occuring in $R(\overline{A})$ and $R(\overline{x},\lambda,s)$ denotes the principal series representation of $\overline{G}_{\mathbb{A}}$ induced from the representation $\overline{\lambda}(\overline{a})e^{sH(\overline{a})}$ of $\overline{A}_{\mathbb{A}} = \overline{A}^1_{\mathbb{A}} \times A^0_\infty$ extended to $\overline{B}_{\mathbb{A}}$ in the obvious way.

Now suppose $\overline{\pi}$ is an irreducible representations of $\overline{G}_{\mathbb{A}}$ which occurs discretely in $\overline{T}$ outside the space of cusp forms. Then $\overline{\pi}$ is a constituent of $\overline{L}^2_0(E)$. From the theory of Eisenstein series for the metaplectic group (see [10]) it follows that such a representation imbeds (as a quotient) in some $R(\overline{x},\lambda,s)$ with $0 < s \leq 1/2$. Thus the relevance of these principal series representations to the problem posed at the end of Subsection 3.3 is obvious.

To analyze $R(\overline{x},\overline{\lambda},s)$ it is necessary to decompose $\overline{\lambda}$ as a "product" of local representations $\overline{\lambda}_v$. However, $\overline{A}_{\mathbb{A}}^1$ is not abelian. Thus it should not seem surprising that each $\overline{\lambda}_v$ has dimension greater than one and $\overline{\lambda}$ is infinite dimensional. So to make sense out of the product

$$\overline{\lambda} = \otimes \overline{\lambda}_v \tag{3.21}$$

we must (as in Section 3.2) interpret it in terms of the cocycle representations it determines. This will be possible since each irreducible unitary β_v-representation of A_v will contain the identity representation of $A_v \cap K_v$ exactly once. (We note that λ_v will turn out to be four-dimensional, at least when v is finite and odd.)

The facts needed to make (3.21) precise will appear in Sections 4 and 5. We shall see that each $\overline{\lambda}_v$ is induced from a character of a subgroup $\overline{B}_v'$ of index four in $\overline{B}_v$. From this it follows that

$$R(\overline{x},\overline{\lambda},s) = \otimes_v \overline{\rho}_v(\overline{x}_v,\mu_1,\mu_2)$$

with

$$\overline{\rho}_v(\overline{x}_1,\mu_1,\mu_2) = \mathrm{Ind}(\overline{B}_v',\overline{G}_v,\zeta\mu_1\mu_2)$$

and μ_1,μ_2 quasi-characters of F_v^x. It is precisely principal series representations of this type which will be discussed in Sections 5 and 6.

Let us return now to $\overline{L}_0^2(E)$. As already noted, the poles of $E(\overline{x},\emptyset,s)$ to the right of $\mathrm{Re}(s) > 0$ all occur at $s = 1/2$. Thus the local components of the corresponding constituents $\overline{\pi}$ of $\overline{L}_0^2(E)$ will be quotients of the local representations $\overline{\rho}_v(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1}(x) = |x|^{1/2}$. Moreover, the global map

$$S: \overline{\pi} \to \pi$$

alluded to in Section 1 will "explain" this phenomenon as follows.

The map S will eventually be defined for irreducible unitary representations of $\overline{G}_{\mathbb{A}}$ non-trivial on Z_2. When $F = \mathbb{Q}(\sqrt{-d})$, S will associate to the representation $\otimes\overline{\pi}_v$ generated by the Eisenstein series (3.18) the one-dimensional representation of $G_{\mathbb{A}}$ generated by the poles of Eisenstein series on $G_{\mathbb{A}}$. For finite v, $S(\overline{\pi}_v)$ will be the class 1 quotient of a principal series representation of index $s = 1$. The corresponding eigenvalues for the zonal spherical function of $\overline{\pi}_v$ will be consistent with those of $S(\overline{\pi}_v)$ and the eigenvalues of $E(u,s)$ with respect to Hecke's ring. On the other hand, if $F = \mathbb{Q}$, suppose the infinite component of $\overline{\pi}$ is a representation of $\overline{G}_{\mathbb{R}}$ whose lowest weight is $k/2 (k \geq 3)$. Then the infinite component of $S(\overline{\pi})$ will be a discrete series representation of $G_{\mathbb{R}}$ whose lowest weight is $k - 1$. Thus the map S will also play a crucial role in explaining Shimura's theory.

Actually, the correspondence for $F = \mathbb{Q}$ will not <u>exactly</u> contain Shimura's (cf. Section 1). Indeed Shimura's forms are automorphic with respect to $\Gamma_0(N)$ and such forms need to be lifted above $\overline{G}_{\mathbb{A}}$ to a slightly larger cover. (A four-fold cover will do since the functional equation for $\theta(z)$ implies that for γ_1, γ_2 in $\Gamma_0(N)$,

$$\frac{J^*(\gamma_1\gamma_2, i)}{J^*(\gamma_1, \gamma_2 i) J^*(\gamma_2, 1)} = \frac{\chi^*(\gamma_1\gamma_2)}{\chi^*(\gamma_1)\chi^*(\gamma_2)}$$

with

$$\chi^*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(\frac{c}{d}\right) \varepsilon_d$$

and $\varepsilon_d = 1$ or i according if d is congruent to 1 or 3 (modulo 4). Thus our results at best are "consistent" with Shimura's.

As already suggested, all these questions will be described more carefully in Section 6 after the necessary facts from representation theory have been obtained.

§4. Local Theory: the Archimedean places.

4.1. Basic Representation theory.

We start by developing the representation theory of $\overline{GL}_2(\mathbb{R})$ in a form suitable for our definition of the local map $S(\overline{\pi}_v)$. The theory for $GL_2(\mathbb{C})$ is well-known and will be recalled briefly in the next Subsection.

Section 3 suggests that we restrict our attention to representations of $\overline{GL}_2(\mathbb{R})$ which are trivial on Z_∞^0, i.e. to representations of the subgroup

$$\overline{SL}_2(\mathbb{R})^* = \{(g,\zeta) \in \overline{GL}_2(\mathbb{R}): \det(g) = \pm 1\}.$$

We start by dealing with $\overline{SL}_2(\mathbb{R})$ which (for the moment only) we denote by $\overline{G}$. Important subgroups are

$$N = \{([\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}], 1)\}$$

$\overline{A}$ = inverse image of $A = \{[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}]\}$ in $\overline{G}$.

$$A_0 = \{[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}] \in A: a > 0\}$$

$\overline{K}$ = inverse image of $SO(2) = K$

and

$\overline{M}$ = centralizer of A_0 in K.

Recall that $\overline{K}$ may be identified with the group of rotations $\overline{r}(\theta) = [\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}]$, with $0 \leq \theta < 4\pi$. $\overline{M}$ is the inverse image of $M = \pm 1\!\!1_2$ and hence is a cyclic group of order 4 generated by $\gamma = \{-1\!\!1_2, 1\}$.

Since A_0 obviously imbeds as a subgroup of $\overline{G}$ (the Hilbert symbol being trivial on $\mathbb{R}_x^+$) the Iwasawa decomposition for $\overline{G}$ is

$$\overline{G} = N\, A_0\, \overline{K}. \tag{4.1}$$

Also, since $\overline{G}$ is a connected semi-simple Lie group with finite center, Harish-Chandra's general theory is completely applicable. In particular, every irreducible unitary representation of $\overline{G}$ is admissible (its restriction to $\overline{K}$ decomposes with finite multiplicities) and every such representation is a subquotient of some (possibly non-unitary) principal series representation of the following type.

Let $\overline{B}$ denote the inverse image in $\overline{G}$ of the upper triangular subgroup of G. Then

$$\overline{B} = \overline{M} A_0 N. \tag{4.2}$$

Fix a quasi-character λ of A_0, an irreducible representation τ of $\overline{M}$, and consider the representation of $\overline{B}$ (actually $\overline{B}/N$) defined by

$$\lambda\otimes\tau(\overline{m}\ a\ n) = \tau(\overline{m})\lambda(a). \tag{4.3}$$

The resulting Banach space representation

$$\rho(\lambda,\tau) = \mathrm{Ind}(\overline{B},\overline{G},\ \lambda\otimes\tau)$$

will be unitary iff λ is, so in general, $\rho(\lambda,t)$ is a non-unitary <u>principal series representation</u>.

The space of $\rho(\lambda,\tau)$, call it $B(\lambda,\tau)$, consists of measurable functions f on $\overline{G}$ such that

$$f(\overline{b}\ \overline{g}) = \delta^{1/2}(\overline{b})\lambda\otimes\tau(\overline{b})f(\overline{g}) \tag{4.4}$$

for $\overline{b}\in\overline{B}$, and

$$\int_{\overline{K}} |f(\overline{k})|^2 d\overline{k} < \infty .$$

Here $\delta(\overline{b}) = \delta([\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}],\zeta) = |a|^2$ denotes the modular function for B. Since $\overline{G} = N A_0 \overline{K}$, $B(\lambda,\tau)$ may also be described as the space of square integrable functions on $\overline{K}$ satisfying

$$f(\overline{b}\ \overline{k}) = \tau(\overline{b})f(\overline{k})$$

for $\overline{b}$ in (the finite group) $\overline{B} \cap \overline{K} = \overline{M}$. Note that $\rho(\lambda,\tau)$ will be trivial on Z_2 (and hence define an ordinary representation of $SL_2(\mathbb{R})$) iff τ is trivial on Z_2. Thus we want to further analyze these representations τ.

There are precisely four irreducible (unitary) representations of $\overline{M}$ and they are given by the characters

$$\tau(\gamma^j) = e^{\pi i j \tau} \tag{4.5}$$

where $j = 1,2,3,4$, and $\tau = 0, 1, 1/2$, or $-1/2$. Since $\gamma^2 = \{-1_2, 1\}$ τ (and hence $\rho(\lambda,\tau)$) factors through Z_2 if and only if $\tau = 0$ or 1. Before describing how $\rho(\lambda,\tau)$ reduces (if at all) we must introduce a convenient basis for $B(\lambda,\tau)$.

If $n a \in NA_0$, and $0 \leq \theta < 4\pi$, define $\varphi_m(\overline{g})$ on $\overline{G}$ by

$$\varphi_m(n a \overline{r}(\theta)) = \lambda \otimes \tau(a) e^{im\theta} \quad . \tag{4.6}$$

Here m is an integer of half-integer depending on τ. More precisely,

(a) m is an even integer if $\tau = 0$;

(b) m is an odd integer if $\tau = 1$;

(c) $m = \ell/2$, with $\ell \equiv 1(4)$ if $\tau = 1/2$; and

(d) $m = \ell/2$, with $\ell \equiv 3(4)$ if $\tau = -1/2$.

In any case, the appropriate collection $\{\varphi_m\}$ comprises a basis for $B(\lambda,\tau)$.

Since every quasi-character of A_0 is of the form $\lambda([\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}]) = a^s$, λ will henceforth be treated as a complex number, namely s. Also, k will denote an odd positive integer greater than or equal to three.

Lemma 4.1. (a) $\rho(\lambda,0)$ is irreducible if λ is not an odd integer. However, if $\lambda = k - 2$, $B(\lambda,0)$ contains exactly two invariant subspaces, namely

$$B_-(\lambda+1) = B_-(k-1) = \sum_{\substack{m \geq \lambda+1 \\ m \equiv \lambda+1(2)}} \mathbb{C}\varphi_m$$

and

$$B_+(\lambda+1) = B_+(k-1) = \sum_{\substack{m\leq -(\lambda+1) \\ m\equiv(\lambda+1)(2)}} \mathbb{C}\varphi_m \quad ;$$

if $\lambda = -(k-2)$, $B(\lambda,0)$ contains exactly one invariant subspace, namely

$$B(\lambda+1) = \sum_{\substack{|m|<-\lambda+1 \\ m\equiv(\lambda-1)(2)}} \mathbb{C}\varphi_m,$$

and this representation is equivalent to the quotient $B(\lambda,0)/B_+(k-1)\oplus B_-(k-1)$. The results for $\rho(\lambda,1)$ are identical with $\lambda = k-3$ (an even integer) replacing $\lambda = k-2$;

(b) $\rho(\lambda,\pm 1/2)$ is <u>irreducible</u> if $\lambda \neq \pm \frac{k-2}{2}$. However, if $\lambda = \frac{k-2}{2}$, <u>with $k \equiv 1(4)$</u>, $B(\lambda,1/2)$ contains exactly one invariant subspace, namely

$$B_-^{1/2}(\lambda+1) = \sum_{\substack{m'\geq k \\ m'\equiv k(4)}} \mathbb{C}\varphi_{m'/2}$$

and $B(\lambda,-1/2)$ contains the invariant subspace

$$B_+^{-1/2}(-\lambda-1) = \sum_{\substack{m'\leq -k \\ m'\equiv -k(4)}} \mathbb{C}\varphi_{m'/2} \cdot$$

On the other hand, if $\lambda = \frac{k-2}{2}$ <u>with $k \equiv 3(4)$</u>, $B(\lambda,1/2)$ contains the invariant subspace $B_+^{1/2}(-k/2)$ and $B(\lambda,-1/2)$ contains $B_-^{-1/2}(k/2)$. Similarly, <u>if $\lambda = -\frac{(k-2)}{2}$</u>, with $k \equiv 3(4)$, then

$B(\lambda,1/2)$ contains $B_-^{1/2}(-k/2+2)$ and $B(\lambda,-1/2)$ contains $B_+^{-1/2}(k/2-2)$; if $\lambda = -\frac{(k-2)}{2}$, and $k \equiv 1(4)$, then $B(\lambda,1/2)$ contains the invariant subspace $B_+^{1/2}(k/2-2)$ and $B(\lambda,-1/2)$ contains $B_-^{1/2}(-\frac{k}{2}+2)$.

(c) The subspaces $B_{\pm}^{\pm 1/2}(\pm(\lambda+1))$ have infinite codimension in $B(\lambda,\pm 1/2)$ and the representation theory at λ is complementary (in the obvious sense) to the theory at $-\lambda$; for example, the irreducible representation of $\overline{G}$ in $B_{-}^{1/2}(1/2)$ (corresponding to $\lambda = -1/2$) is equivalent to the quotient representation $B(-1/2,1/2)/B_{+}^{1/2}(3/2)$ (corresponding to $\lambda = 1/2$).

Proof. Part (a) concerns $SL_2(\mathbb{R})$ and is well-known. Since the Lie algebra argument used to prove it works equally well for the covering group we simply sketch the argument below.

Consider first the elements

$$V_+ = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad V_- = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

in the complexification of the Lie algebra of $\overline{G_{\mathbb{R}}}$. These elements clearly span the complexification. Furthermore, they act on $B(\lambda,\tau)$ through the formula

$$\rho(X)f(g) = \frac{df}{dt}\,(g\,\exp(tX))\Big|_{t=0},$$

valid for smooth f in $B(\lambda,\tau)$ and arbitrary X in the Lie algebra (hence also for X in the complexification by the obvious extension). Thus one obtains a representation $\rho(X)$ of the complexified Lie algebra (hence also the universal enveloping algebra) in the space of K-finite functions in $B(\lambda,\tau)$.

For each basis element of $B(\lambda,\tau)$,

$$\rho(U)\varphi_m = im\varphi_m \tag{4.7}$$

and

$$\rho(V_{\pm})\varphi_m = (\lambda+1\pm m)\varphi_{m\pm 2}\ . \tag{4.8}$$

These formulas are independent of τ. Moreover, the first implies

each invariant subspace of $B(\lambda,\tau)$ is spanned by the φ_m it contains. Therefore, since $\rho(\lambda,\tau)$ is (topologically) irreducible if and only if $\rho(X)$ is algebraically irreducible, the Lemma follows from (4.8). □

Lemma 4.2. The following representations exhaust a set of representatives for the equivalence classes of irreducible unitary representations of $\overline{G}$:

(a) The representations $\rho(it,\tau)$ with $t \in \mathbb{R}$;

(b) The representations $\rho(s,0)$ with $-1 < s < 1$, $s \neq 0$; the representations $\rho(s,\pm 1/2)$ with $-1/2 < s < 1/2$, $s \neq 0$;

(c) The representations in $B_{\pm}(\overline{\mp}(k-1))$, $B_{\pm}(\overline{\mp}(k-2))$, $B_{-}^{1/2}(\frac{k}{2})$, $B_{+}^{-1/2}(-k/2)$, $B_{+}^{1/2}(-k/2)$, and $B_{-}^{-1/2}(k/2)$ with $k \geq 3$;

(d) The trivial representation of $SL_2(\mathbb{R})$, and the representations of $\overline{G_{\mathbb{R}}}$ in $B_{-}^{1/2}(1/2)$ and $B_{+}^{-1/2}(-1/2)$.

Proof. The results for $SL_2(\mathbb{R})$ are again well-known and the results for $\overline{SL}_2(\mathbb{R})$ can be deduced from Sally ([33] and [34]). In all cases except (a) one must introduce a new inner product in $B(\lambda,\tau)$ (or an appropriate subspace thereof). Also, to be precise, we must recall that $\rho(0,1)$ is reducible and the direct sum of two irreducible representations. □

Terminology.

The representations $\rho(\lambda,\tau)$ in (a) will be called continuous series representations, those in (b) complementary series representations. The only equivalences are $\rho(\lambda,\tau) = \rho(-\lambda,\tau)$.

The representations in (c) will be called discrete series representations of lowest (or highest) weight $k-1$ (or $-(k-1)$) (or $k/2$), etc., whichever is appropriate. They will be denoted by π_{k-1}^{-} (or π_{k-1}^{+}, or $\overline{\pi}_{k/2}^{-}$,...) to emphasize that among the characters of K appearing in them, the lowest is $e^{i(k-1)\theta}$. (The highest in π_{k-1}^{+} is $e^{-i(k-1)\theta}$, the lowest in $\overline{\pi}_{k/2}^{-}$ is $e^{ik/2\,\theta}$,

and so on.) All these representations are square-integrable on $\overline{G}$, and all but $\pi_2^{\pm}$ and $\pi_{3/2}^{\pm}$ are actually integrable.

The representations in (d) will be denoted $\overline{\pi}_{1/2}^{\pm}$. $\overline{\pi}_{1/2}^{-}$ has lowest weight $1/2$ and $\overline{\pi}_{1/2}^{+}$ has highest weight $-1/2$. The character of $\pi_{1/2}^{\pm}$ is non-tempered but does not vanish on the elliptic set* of $\overline{SL_2}(\mathbb{R})$.

Now we return to the group $\overline{G^*}$, the inverse image in $\overline{GL_2}(\mathbb{R})$ of $\{g \in GL_2(\mathbb{R}): \det(g) = \pm 1\}$. To describe its representation theory it is convenient to appeal to (a special case of) Mackey's generalization of Clifford's theory of representations induced from a normal subgroup. We recall this theory here in an abstract setting since we shall use it again in another context in Section 5.

Suppose N is a normal subgroup of index two in the locally compact group G. If h is a non-trivial coset representative of G/N, and σ is a representation of N, then σ is said to be <u>self-conjugate</u> if it is equivalent to its "conjugate" representation

$$\sigma^h(n) = \sigma(n^h) = \sigma(hnh^{-1}).$$

From Mackey [26] one can deduce the following:

<u>Lemma 4.3</u>. Suppose N and H are as above, and σ is unitary and irreducible. Then

(a) If σ is <u>not</u> self-conjugate, $\mathrm{Ind}(N,H,\sigma)$ is irreducible and equivalent to $\mathrm{Ind}(N,H,\sigma^h)$;

(b) If σ <u>is</u> equivalent to σ^h, $\mathrm{Ind}(N,H,\sigma)$ is the sum of two irreducible representations whose restrictions to N coincide with σ;

(c) All irreducible unitary representations of G arise from N as above (assuming everything in sight is type 1).

To apply this Lemma to $\overline{G^*}$ we fix h equal to $([\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}],1)$.

The crucial facts are then included in:

*Such representations are called "trash" by P. Sally and N. Wallach.

Lemma 4.4. (a) The self-conjugate representations of $SL_2(\mathbb{R})$ are the continuous, complementary, and trivial representations; the conjugate of π_k^+ is π_k^-;

(b) None of the genuine representations of $\overline{SL_2}(\mathbb{R})$ is self-conjugate; in fact, each is conjugate to its obvious mate of opposite $\overline{M}$-type: for example, $\rho(it, 1/2)$ is conjugate to $\rho(it, -1/2)$, $\pi_{3/2}^+$ is conjugate to $\pi_{3/2}^-$, and so on.

Proof. Part (a) is proved in Section 2.1 of Gelbart [8]. For (b), start with the identity

$$h\left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix}, \zeta\right)h^{-1} = \left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix}, \operatorname{sgn}(a)\zeta\right). \tag{4.9}$$

(This follows from Lemma 2.11 and the definition of the Hilbert symbol for $\mathbb{R}$.) Now for each f in $B(\lambda,1/2)$, put $f^h(\overline{g}) = f(h\overline{g})$. Clearly f^h is zero iff f is, and

$$\begin{aligned} f^h(\overline{b}\ \overline{g}) &= f(h\ \overline{b}\ \overline{g}) = f(\overline{b}^h h\ \overline{g}) \\ &= \delta^{1/2}(\overline{b})\lambda\otimes\tau_{1/2}(\overline{b}^h) f^h(\overline{g}). \end{aligned}$$

But by (4.9), $\lambda\otimes\tau(\overline{b}^h) = \lambda\otimes\tau_{-1/2}(\overline{b})$, so the map $f \to f^h$ intertwines $\rho(\lambda,1/2)$ with $\rho(\lambda,-1/2)$. Since it also takes invariant subspaces to invariant subspaces, the proof is complete. □

Proposition 4.5. Every irreducible unitary representation of G^* belongs to one of the following series of representations:

(a) the continuous series of representations

$$\rho(\lambda_\varepsilon, \eta) = \operatorname{Ind}(B^*, G^*, \lambda_\varepsilon^\eta)$$

where $B^* = \left\{\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}\right\}$ is the full triangular subgroup of $G^* = \{g \in GL_2(\mathbb{R}): \det(g) = \pm 1\}$, $\lambda \in i\mathbb{R}$, ε and $\eta = 0$ or 1, and

$$\lambda_\varepsilon^\eta\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix}\right) = |a_1|^\lambda [\operatorname{sgn}(a_1)]^\varepsilon [\operatorname{sgn}(a_2)]^\eta \ ;$$

(b) the complementary series $\rho(\lambda_0,\eta)$ where λ is between 0 and 1;

(c) the discrete series representations

$$\pi_{k-1} = \mathrm{Ind}(G,G^*,\pi^+_{k-1}) = \mathrm{Ind}(G,G^*,\pi^-_{k-1})$$

and π_{k-2}, where $k \geq 3$ is odd;

(d) the one-dimensional representations $g \to [\det(g)]^\epsilon$, $\epsilon = 0$ or 1.

Proposition 4.6. Every irreducible unitary representation of $\overline{G^*}$, non-trivial on Z_2, belongs to one of the following series of representations:

(a) the continuous series representations

$$\rho(\lambda) = \mathrm{Ind}(\overline{B},\overline{G^*},\lambda\otimes\tau_{1/2}) = \mathrm{Ind}(\overline{B},\overline{G^*},\lambda\otimes\tau_{-1/2})$$

where $\lambda \in i\,\mathbb{R}^\times_+$;

(b) the complementary series representations $\rho(\lambda)$ where $0 < \lambda < 1/2$;

(c) The discrete series representations

$$\overline{\pi}_{k/2} = \mathrm{Ind}(\overline{G},\overline{G^*},\overline{\pi}^+_{k/2})$$

$$= \mathrm{Ind}(\overline{G},\overline{G^*},\overline{\pi}^-_{k/2})$$

(d) the representation $\overline{\pi}_{1/2} = \mathrm{Ind}(\overline{G},\overline{G^*},\overline{\pi}^\pm_{1/2})$.

Proof of Proposition 4.5. Observe that

$$\mathrm{Ind}(G,G^*,\rho(\lambda,\epsilon)) = \mathrm{Ind}(G,G^*,\mathrm{Ind}(B,G,\lambda_\epsilon))$$

$$= \mathrm{Ind}(B,G^*,\lambda_\epsilon) = \mathrm{Ind}(B^*,G^*,\mathrm{Ind}(B,B^*,\lambda_\epsilon))$$

where $\lambda_\epsilon(\begin{bmatrix} a & x \\ 0 & a^{-1}\end{bmatrix}) = |a|^\lambda[\mathrm{sgn}(a)]^\epsilon$. But $\mathrm{Ind}(B,B^*,\lambda_\epsilon) = \lambda^0_\epsilon \oplus \lambda^1_\epsilon$.

Thus the Proposition follows from Lemmas 4.2, 4.3, and 4.4. □

Proof of Proposition 4.6. Apply Lemmas 4.2 through 4.4 directly.

4.2. The Local Map.

As in Subsection 4.1, G will denote $SL_2(\mathbb{R})$, G^* the group $\{g \in GL_2(\mathbb{R}) : \det g = \pm 1\}$, and $\overline{G}^*$ its inverse image in $\overline{GL}_2(\mathbb{R})$.

From Section 3, and the work of Shirmua, the existence of a non-trivial correspondence associating representations of $SL_2(\mathbb{R})$ to genuine representations of $\overline{G}^*$ is clearly indicated. It is also indicated that this correspondence should associate π_{k-1} to $\overline{\pi}_{k/2}$. In this paragraph we shall define such a correspondence in very simple terms. In the next we shall construct (part of) an inverse to this correspondence in completely different terms.

We begin by setting up a one-to-one correspondence between certain ordered pairs of quasi-characters (μ_1,μ_2) of $\mathbb{R}^x$ and certain irreducible unitary representations of $\overline{G}^*$.

Suppose first that $\overline{\pi}$ is an irreducible unitary representation of $\overline{G}^*$ non-trivial on Z_2. Let μ_1,μ_2 be quasicharacters of $\mathbb{R}^x$ satisfying the following properties:

(1) $\mu_1(-1) = \mu_2(-1) = 1$;

(2) $\mu_1 = \mu_2^{-1}$ on $\mathbb{R}_+^x$; and

(3) $\mu_1\mu_2^{-1}(x) = |x|^s$ with $\mathrm{Re}(s) \geq 0$, $\mathrm{Im}(s) \geq 0$.

Let $\overline{\rho}(\mu_1,\mu_2)$ denote the representation of $\overline{G}^*$ induced from the character

$$\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix},\zeta\right) \to \zeta\mu_1(a_1)\mu_2(a_2)$$

of $\overline{B}^* = \{\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix},\zeta\right) : a_1a_2 = \pm 1\}$.

By Proposition 4.6 there will be exactly one pair (μ_1,μ_2) satisfying (1)-(3) and such that some (possibly trivial) subquotient of $\overline{\rho}(\mu_1,\mu_2)$, call it $\overline{\pi}(\mu_1,\mu_2)$, is equivalent to $\overline{\pi}$. Hence we can speak of $\overline{\pi}$ and $\overline{\pi}(\mu_1,\mu_2)$ in one and the same breath.

Now we want to associate to each such pair (μ_1,μ_2) an irreducible unitary representation of G^*. So let $\rho(\mu_1,\mu_2)$

denote the representation of G^* induced from the character

$$\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} \to \mu_1(a)\mu_2(a_2)$$

of B^*. If $\mu_i(x) = |x|^{\lambda_i}$, then $\rho(\mu_1,\mu_2)$ is equivalent to the representation $\rho(\lambda_0,0)$ of Proposition 4.5 with $\lambda = \lambda_1-\lambda_2$. Thus we can associate to each (μ_1,μ_2) satisfying (1)-(3) above a well-defined irreducible representation of G^*, namely $\rho(\mu_1,\mu_2)$ itself if it is irreducible, or "the" irreducible unitary representation defined in a subquotient of $\rho(\mu_1,\mu_2)$ if $\rho(\mu_1,\mu_2)$ is reducible. We shall denote this representation by $\pi(\mu_1,\mu_2)$. The only ambiguity is when $\lambda = 1$ (in which case $\pi(\mu_1,\mu_2)$ is either the trivial representation or the discrete series representation π_2).

Definition 4.7. Let $\overline{\Lambda}$ denote the collection of equivalence classes of irreducible unitary genuine representations of $\overline{G}^*$. If $\overline{\pi} \in \overline{\Lambda}$ corresponds to (μ_1,μ_2), define $S(\overline{\pi})$ to be $\pi(\mu_1^2,\mu_2^2)$. Here $\pi(\mu_1^2,\mu_2^2)$ is associated to the pair (μ_1^2,μ_2^2) according to the following convention: if $\overline{\pi}(\mu_1,\mu_2)$ is a subrepresentation (resp. quotient) of $\overline{\rho}(\mu_1,\mu_2)$, then $\pi(\mu_1^2,\mu_2^2)$ is a subrepresentation (resp. quotient) of $\rho(\mu_1^2,\mu_2^2)$.

Note that $\rho(\mu_1^2,\mu_2^2)$ will be reducible if and only if $\overline{\rho}(\mu_1,\mu_2)$ is reducible.

Proposition 4.8. The map

$$S:\overline{\pi} \to \pi$$

is one-to-one from $\overline{\Lambda}$ onto the collection of irreducible unitary representations of G^* which are either class 1 or discrete series representations of even weight. Moreover, if $k \geq 3$,

$$S(\overline{\pi}_{k/2}) = \pi_{k-1}$$

and the inverse image under S of the trivial representation of G^* is the non-tempered representation $\overline{\pi}_{1/2}$.

Proof. The proof is immediate from the representation theory thus far developed. In fact our definition of the local map was inspired by the prerequisite that this Proposition is true. □

The definition of the local map for $F = \mathbb{C}$ is analogous but simpler since $\overline{GL}_2(\mathbb{C})$ is the direct product of $GL_2(\mathbb{C})$ with Z_2.

Let μ_1,μ_2 denote quasi-characters of $\mathbb{C}^x$ and $\rho(\mu_1,\mu_2)$ the representation of $GL_2(\mathbb{C})$ induced from the character $\mu_1\mu_2$ of its upper triangular subgroup. Let $B(\mu_1,\mu_2)$ denote the space of $\rho(\mu_1,\mu_2)$ and $B(\rho_n)$ the subspace which transforms according to the unique representation of $SU_2(\mathbb{C})$ of dimension $n+1$. All the representation theory we shall need for $GL_2(\mathbb{C})$ is contained in the two results below. The appropriate references are Section 6 of [18] and Chapter 3 of [12].

Proposition 4.9.

(a) $\rho(\mu_1,\mu_2)$ is irreducible if $\mu_1\mu_2^{-1}(z) \neq z^{p}\bar{z}^{q}$ with $p \geq 1$, $q \geq 1$; in this case, $\rho(\mu_1,\mu_2)$ is denoted $\pi(\mu_1,\mu_2)$;

(b) If $\mu_1\mu_2^{-1}(z) = z^{p}\bar{z}^{q}$ with $p \geq 1$, $q \geq 1$,

$$B_s(\mu_1,\mu_2) = \sum_{\substack{n \geq p+2 \\ n \equiv p+q(2)}} B(\rho_n)$$

is the only proper invariant subspace of $B(\mu_1,\mu_2)$; in this case, $\sigma(\mu_1,\mu_2)$ will denote the representation of $GL_2(\mathbb{C})$ in $B_s(\mu_1,\mu_2)$ and $\pi(\mu_1,\mu_2)$ the representation in $B/B_s(\mu_1,\mu_2)$.

(c) If $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$, with $p \geq 1$, $q \geq 1$

$$B_f(\mu_1,\mu_2) = \sum_{\substack{|p-q| \leq n < p+q \\ n \equiv p+q(2)}} B(\rho_n)$$

is the only proper invariant subspace of $B(\mu_1,\mu_2)$; in this case, $\pi(\mu_1,\mu_2)$ will denote the representation in $B_f(\mu_1,\mu_2)$ and $\sigma(\mu_1,\mu_2)$ the representation in the quotient $B/B_f(\mu_1,\mu_2)$.

(d) $\pi(\mu_1,\mu_2)$ is equivalent to $\pi(\mu_1',\mu_2')$ if and only if $(\mu_1,\mu_2) = (\mu_1',\mu_2')$ or (μ_2', μ_1');

(e) If $\mu_1\mu_2^{-1}(z) = z^p\bar{z}^q$, with $p \geq 1$ and $q \geq 1$, there is a pair (ν_1,ν_2) such that $\nu_1\nu_2 = \mu_1\mu_2$ and $\sigma(\mu_1,\mu_2) = \pi(\nu_1,\nu_2)$; finally,

(f) Every irreducible admissible representation of $GL_2(\mathbb{C})$ is a $\pi(\mu_1,\mu_2)$ for some choice of (μ_1,μ_2).

Proposition 4.10. Each irreducible unitarizable representation of $GL_2(\mathbb{C})$ belongs to one of the following series of representation:

(a) the continuous series $\pi(\mu_1,\mu_2)$ with μ_1,μ_2 unitary;

(b) the complementary series representations $\pi(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1}(z) = |z|^{2s}$, $0 < s < 1$;

(c) the one-dimensional representations $g \to \mu(\det(g))$, μ a unitary character of $\mathbb{C}^\times$.

Now let $\bar{\chi}$ denote the non-trivial character of Z_2 and $\bar{\pi}$ an irreducible unitary representation of $\overline{GL2}(\mathbb{C})$ non-trivial on Z_2. Then $\bar{\pi} = \bar{\chi} \otimes \pi(\mu_1,\mu_2)$ for some pair of quasi-characters μ_1,μ_2 of $\mathbb{C}^\times$. Moreover, by Part (d) of Proposition 4.9, μ_1 and μ_2 are completely determined by $\bar{\pi}$.

Definition 4.11. Let $\bar{\Lambda}$ denote the collection of irreducible unitary representations of $GL_2(\mathbb{C})$ non-trivial on Z_2. If $\bar{\pi} \in \bar{\Lambda}$ corresponds to (μ_1,μ_2) as above, define $S(\bar{\pi})$ to be $\pi(\mu_1^2,\mu_2^2)$.

Note that the correspondence

$$S:\bar{\pi} \to \pi$$

takes continuous series representations to continuous series representations and class 1 representations to class 1 representations. It also takes "the" complementary series representation at $s = 1/2$ (i.e. $\pi(\mu_1,\mu_2)$, with $\mu_1\mu_2 = 1$ and $\mu_1\mu_2^{-1}(z) = |z|^2$) to the trivial representation. (In this sense, the complementary series representation at $s = 1/2$ attempts to be a "non-tempered" repre-

sentation of the covering group; cf. Kubota [22], [23], and Section 6 of this paper.

Note, finally, that $S(\overline{\pi})$ should be unitary whenever $\overline{\pi}$ is. Indeed the only possible exception is when $\overline{\pi}$ is a complementary series representation of index $1/2 < s < 1$. This possibility, at least in the context of cusp forms, should be eliminated by global (L-function) considerations.

4.3 Weil's Representation in 3-variables.

Let V denote the 3-dimensional space of 2×2 real symmetric matrices. We shall identify V with the Lie algebra of $SL_2(R)$ via the map

$$\begin{bmatrix} x_1+x_2 & x_3 \\ x_3 & x_1-x_2 \end{bmatrix} \longleftrightarrow (x_1,x_2,x_3) \longleftrightarrow \begin{bmatrix} x_2 & -x_1+x_3 \\ x_1+x_3 & -x_2 \end{bmatrix} = X.$$

Thus a natural basis for V is provided by the matrices

$$\ell_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \ell_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } \ell_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $X = (x_1,x_2,x_3) = \sum_{i=1}^{3} x_i \ell_i$.

Our purpose in this Subsection is to decompose the Weil representation attached to the quadratic form on V defined by $q(X) = \det(X)$, equivalently,

$$q(x_1,x_2,x_3) = x_1^2 - x_2^2 - x_3^2.$$

Call this representation r_q. The decomposition of r_q ties in with the map $\overline{\pi} \to \pi$ because r_q defines a representation of $\overline{SL_2}(\mathbb{R})$ and the orthogonal group of $q(X)$ is (essentially) $SL_2(\mathbb{R})$.

To be more specific, fix $\tau(x) = e^{\pi i x}$. Then the Weil representation attached to (q,V,τ) operates in $L^2(V,dX)$ through the operators

$$r_q\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) \Phi(X) = e^{\pi i b q(X)} \Phi(X) \tag{4.10}$$

and

$$r_q\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \Phi(X) = e^{-\pi i/4} \hat{\Phi}(-X). \tag{4.11}$$

By Corollary 2.24, the representation determined by (4.10) and (4.11) is not an ordinary representation of $SL_2(\mathbb{R})$. Rather it is a β-representation with β cohomologous (in $Z^2(SL_2(\mathbb{R}),Z_2)$) to the factor set described by (2.6). Thus r_q defines a representation

of $\widetilde{SL}_2(\mathbb{R})$ non-trivial on Z_2 .

Now we introduce the orthogonal group of q . This group is the proper Lorentz group in 3-variables (1 time, 2 space variables) usually denoted by $SO(1,2)$. It operates on $L^2(V)$ as a group of unitary operators

$$H(g)\Phi(X) = \Phi(gX)$$

and obviously commutes with r_q since it commutes with the generators (4.10) and (4.11). For this reason one expects to be able to decompose r_q according to the decomposition of the representation $H(g)$.

The group $SL_2(\mathbb{R})$ comes into play as usual through its adjoint action

$$Ad(g)X = gXg^{-1}$$

on V. This action preserves $q(X)$ and the homomorphism $g \to Ad(g)$ provides a surjection of $SL_2(\mathbb{R})$ onto the connected component of $SO(1,2)$ (with kernel $\{\pm \mathbb{1}_2\}$). Thus $SL_2(\mathbb{R})/(\pm \mathbb{1}_2)$ can be identified with a subgroup of $SO(1,2)$; the resulting unitary representation of $SL_2(\mathbb{R})$ in $L^2(V)$, given by

$$\Phi(X) \to H(g)\Phi(X) = \Phi(Ad(g^{-1})X)$$

will be called the regular representation of $SL_2(\mathbb{R})$ in $L^2(V)$.

Our first task is to describe the decomposition of $H(g)$ into irreducible representations. Roughly speaking, if some primary representation of $SL_2(\mathbb{R})$ occurs in $H(g)$, its space will realize a multiple of some irreducible contituent of r_q, hence index an irreducible representation of $\widetilde{SL}_2(\mathbb{R})$. The resulting correspondence

$$\pi \to \overline{\pi}$$

will then be of some interest.

In what follows we shall restrict our attention to representations of the discrete series and simply make parenthetical remarks

for the remaining representations. Thus our task is to describe a subrepresentation of $H(g)$ equivalent to infinitely many copies of $\pi^{\pm}_{k-1}$. Note that $H(g)$ is trivial on $\{\pm I_2\}$, hence its possible constituents, including $\pi^{\pm}_{k-1}$, must also be.

Under the action of $SL_2(\mathbb{R})$, the space V decomposes into orbits of the following type:

(1) the origin (of no interest);

(2) the forward (resp. backward) light cone

$$\{x: q(X) = 0, x_1 \geq 0 \text{ (resp. } x_1 \leq 0)\}$$

(3) a sheet of the hyperboloid of two sheets

$$H^+_m = \{X: q(X) = m^2\} ;$$

(4) the hyperboloid of one sheet

$$H^-_m = \{X: q(X) = -m^2\} .$$

The action of $SL_2(\mathbb{R})$ on each of these orbits is transitive: (2) is isomorphic to $S\ell_2(\mathbb{R})/N$, H^+_1 is isomorphic to $S\ell_2(R)/K$, and

$$H^-_1 = SL_2(R)/A .$$

Our interest will primarily be in H^-_1 since it is in this orbit that discrete series appear.

Let H^- denote the region $\{X: q(X) < 0\}$. Each $X = (x_1, x_2, x_3)$ in H^- is of the form mh with $m = -|q(X)|^{1/2} \in (-\infty, 0)$ and $h = (\frac{x_1}{m}, \frac{x_2}{m}, \frac{x_3}{m}) \in H^-_1$. The corresponding integral formula is

$$\int_{H^-} f(X)dX = \int_{H^-_1} \int_{-\infty}^{0} f(m,h)|m|^2 dm\, dh$$

where dh denotes the invariant measure on H^-_1 inherited from $SL_2(\mathbb{R})$ and dm is Lebesgue measure on $(-\infty, 0)$. Thus

$$L^2(H^-) = L^2((-\infty,0), |m|^2 dm) \otimes L^2(H^-_1) .$$

To decompose the $H(g)$ - invariant subspace $L^2(H^-)$ we need first to decompose $L^2(H^-_1)$.

Henceforth, k is an odd integer greater than or equal to three. Let G_k denote the space of functions $g(X)$ on V such that

(i) $\square g \equiv 0$, where $\square = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$;

(ii) g is homogeneous of degree $\frac{k-3}{2}$, i.e.

$$g(tX) = g(tx_1, tx_2, tx_3) = t^{\frac{k-3}{2}} g(X) \quad \text{for all} \quad t > 0 ;$$

(iii) $g(-X) = (-1)^{\varepsilon} g(X)$ where $\varepsilon = 0$ if $k \equiv 1(4)$ and $\varepsilon = 1$ if $k \equiv 3(4)$; and

(iv) $g(X)$ vanishes outside H^-.

Note that such g are completely determined by their restriction to H_1^-. Thus G_k is a Hilbert space with inner product given by the inner product in $L^2(H_1^-)$.

That G_k realizes an essentially irreducible unitary representation of $SL_2(\mathbb{R})$ can be deduced from recent work of Strichartz ([44]) and Ehrenpreis ([53]). The precise result is:

Proposition 4.12. The natural action of $SL_2(\mathbb{R})$ in G_k realizes the representation $\pi_{k-1}^+ \oplus \pi_{k-1}^-$.

This follows from a careful study of the K-types appearing in [44] (pulled back from $SO(1,2)$ to $SL_2(\mathbb{R})$).

If we let $L^2(k,H^-)$ denote the subspace of functions in $L^2(V)$ which can be approximated by linear combinations of functions of the form

$$G(X) = f(-|q(X)|^{1/2}) g(X) ,$$

with $g \in G_k$ and $f(m) \in L^2((-\infty,0), |m|^{k-1} dm)$, we have:

Corollary 4.13. $L^2(k,H^-)$ is an invariant subspace of $L^2(V)$ for $H(g)$ and (as an $SL_2(\mathbb{R})$-module) is equivalent to infinitely many copies of $\pi_{k-1}^- \oplus \pi_{k-1}^+$.

Now using a series of non-trivial Lemmas we shall establish the following:

Theorem 4.14. $L^2(k,H^-)$ is an invariant subspace of $L^2(V)$ for r_q and (as an $\overline{SL}_2(\mathbb{R})$ - module) is equivalent to infinitely many copies of $\overline{\pi}^-_{k/2}$.

Lemma 4.15. Suppose $G(X) = f(-|q(X)|^{1/2})q(X)$ belongs to $L^2(k,H^-)$ with $f(m) \in \mathcal{S}(\mathbb{R})$ (the Schwartz space of $\mathbb{R}$). Then:

(a) $r_q([\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}])G(X) = f'(-|q(X)|^{1/2})g(X)$

where

$$f'(m) = e^{\pi i b m^2} f(m);$$

(b) $r_q([\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}])G(X) = e^{-\pi i/4}\overline{f}(-|q(X)|^{1/2})g(X)$

where

$$\overline{f}(m) = e^{\pi i \epsilon/2} \int_0^\infty (m^{-1}t)^{\frac{k-2}{2}} \cos \tfrac{\pi}{2}[\tfrac{k-1}{2} + \epsilon]\, J_{\frac{k-2}{2}}(2mt) f(-t) t\, dt .$$

Proof. Part (a) is completely straightforward. Part (b), on the other hand, results from formulas appearing in Strichartz [43] (Theorem 1). Roughly speaking, Strichartz' theorem describes the Fourier transform of a function on V which transforms (under the natural action of $SL_2(\mathbb{R})$ on V) according to a given irreducible representation of G. If this representation has a highest or lowest weight vector k-1 condition (3) in the definition of G_k implies that $(\sigma \pm \epsilon)$ on p.509 of [43] must be odd. Thus the formulas there simplify considerably and an application of them to our situation proves the lemma. □

Corollary 4.16. $L^2(k,H^-)$ is invariant for r_q. Furthermore, the restriction of r_q to $L^2(k,H^-)$ is equivalent to infinitely many copies of the β-representation of $SL_2(\mathbb{R})$ generated by the operators

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : f \to f'$$

and

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : f \to e^{-\pi i/4}\, \overline{f}\ .$$

To complete the proof of Theorem 4.14, observe that $L^2((-\infty,0),|m|^{k-1}d|m|)$ is isomorphic to $L^2((0,\infty),m\,dm)$ via the map

$$f(r) \to f(-r)|r|^{\frac{k-1}{2}}\ .$$

Thus Theorem 4.14 follows from Lemma 4.17 below coupled with some straightforward manipulations. (A useful identity here is the following:

$$\cos\tfrac{\pi}{2}[\tfrac{k-1}{2}+\varepsilon] = e^{-\frac{\pi i}{4}(k-1)}e^{\pi i\varepsilon/2}\ .)$$

Lemma 4.17. The representation $\overline{\pi}^{-}_{k/2}$ (resp. $\overline{\pi}^{+}_{k/2}$) of $\overline{SL}_2(\mathbb{R})$ is realizable in $L^2((0,\infty),m\,dm)$ so that

(a) the operator corresponding to $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ is

$$f(m) \to e^{\pi i b m^2} f(m) \quad (\text{resp. } e^{-\pi i b m^2} f(M))\ ;$$

(b) the operator corresponding to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is

$$f(m) \to e^{-ik\pi/4} \int_0^\infty J_{\frac{k-2}{2}}(2mt) f(t)\, t\, dt \quad (\text{resp. } e^{ik\pi/4})\ .$$

Proof. Explicit realizations of the representations of $\overline{SL}_2(\mathbb{R})$ are to be found in Pukansky [32] and Sally [33]. In particular, $\overline{\pi}^{\pm}_{k/2}$ corresponds to $R^{\pm}(\pm h)$ in [33] with $h = k/4$. Thus our Lemma follows from Lemma 4.1.5 of [33]. □

Corollary 4.18. There is a natural correspondence (the "duality correspondence") which associates to the representation π_{k-1} of G^* the genuine representation $\overline{\pi}_{k/2}$ of $\overline{G}^*$. In particular, this correspondence inverts (part of) the local map $S_{\mathbb{R}}$ of Section 4.2.

Proof. Let r^*_q denote the representation of $\overline{G^*}$ induced from r_q (viewed as a representation of $\overline{G} = \overline{SL}_2(\mathbb{R})$). Although r_q depends on τ, it can be shown that r^*_q does not. (Cf. Lemma 4.3 and the proof of Lemma 4.4.) In particular, r^*_q will contain copies of $\overline{\pi}_{k/2} = \mathrm{Ind}(\overline{G}, \overline{G^*}, \pi^{\pm}_{k/2})$.

So now consider π_{k-1}. Its restriction to G is $\pi^{+}_{k-1} \oplus \pi^{-}_{k-1}$ and this representation indexes the primary component of r_q which when induced up to $\overline{G^*}$ contains infinitely many copies of $\overline{\pi}_{k/2}$. □

We note that one can extend this correspondence to class 1 continuous series representations as well since these representations also occur in the decomposition of $L^2(V)$ (albeit continuously). The required Fourier transform formula is now more complicated than the formula in Lemma 4.15 but again it can be derived from [43]. The result is that $\pi(\mu_1, \mu_2)$ is mapped to $\overline{\pi}(\mu_1^{1/2}, \mu_2^{1/2})$. More comprehensive speculations about how this correspondence inverts $S : \overline{\pi} \to \pi$ will appear in Section 6.

Concluding Remark. The representation r^*_q above is essentially the representation r_3 discussed in Section 2.5. More precisely, let r'_q denote the restriction of r^*_q to $\overline{SL}_2(\mathbb{R})$ tensored with the trivial representation of $\overline{SL}_2(\mathbb{R})$ in $L^2((0,\infty))$. Then r'_q is isomorphic to $\int_{\mathbb{R}^x} r_3(\tau_t)\,dt$.

4.4 The basic Weil representation (joint with P. Sally)

The Weil correspondence discussed in Corollary 4.18 pairs together discrete series representations of $GL_2(\mathbb{R})$ trivial on Z_∞ with genuine discrete series representations of $\overline{GL}_2(\mathbb{R})$ trivial on Z_∞^0. Its domain actually extends to a larger set. Consider all irreducible "tempered" representations of $GL_2(\mathbb{R})$ trivial on $\pm I$. Equivalently, consider all possible constituents of the "adjoint" representation of $GL_2(\mathbb{R})$ in $L^2(\mathbb{R}^3 \times \mathbb{R}^\times)$. The correspondence alluded to above pairs together constituents of this adjoint representation with constituents of r_3.

Our purpose in this Subsection is to describe a similar (but simpler) correspondence relating a subset of the dual group of $\overline{GL}_2(\mathbb{R})$ to the dual group of $GL_1(\mathbb{R})$. In place of r_3 we deal with r_1. Actually, for convenience, we deal with the representation $r_1(\tau)$ of $\overline{SL}_2(\mathbb{R})$. Thus we deal with $\overline{GL}_2(\mathbb{R})$ modulo its center and $GL_1(\mathbb{R})$ modulo $(0,\infty)$.

Fix $\tau(x) = e^{\pi i x}$. Then the action of $r_1(\tau)$ in $L^2(\mathbb{R})$ is given by the operators

$$r_1(\tau)(\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right])\Phi(x) = e^{\pi i b x^2}\Phi(x)$$

and

$$r_1(\tau)(\overline{w})\Phi(x) = e^{\pi i/4}\,\hat{\Phi}(x).$$

Theorem 4.19.

(a) With the notation of Subsection 4.1,

$$r_1(\tau) = \bar{\pi}_{1/2}^- \oplus \bar{\pi}_{3/2}^- .$$

Recall that $\bar{\pi}_{1/2}^-$ is a subquotient of the principal series representation $\rho(1/2,1/2)$; it has lowest weight $1/2$; $\bar{\pi}_{3/2}^-$ is a subrepresentation of $\rho(1/2,-1/2)$; it has lowest weight vector $-3/2$.

(b) Let $r_1(\tau_{-1})$ denote the Weil representation corresponding to $\tau_{-1}(x) = e^{-\pi i x}$. Then

$$r_1(\tau_{-1}) = \overline{\pi}^+_{1/2} \oplus \overline{\pi}^+_{3/2} .$$

Here $\overline{\pi}^+_{1/2}$ has highest weight vector $-1/2$ and $\overline{\pi}^+_{3/2}$ has highest weight vector $3/2$.

Proof (1) (Sketch). It is easy to see that the subspace of $L^2(\mathbb{R})$ consisting of even (resp. odd) functions is irreducibly invariant for $r_1(\tau_{\pm 1})$. To identify the resulting irreducible representations of $\overline{SL}_2(\mathbb{R})$ one uses the models for $\overline{\pi}^{\pm}_{k/2}$ presented in Chapter IV of [33]. □

Proof (2) (Sketch). Let r_1^e (resp. r_1^0) denote the restriction of r_1 to the space of even (resp. odd) functions in $L^2(\mathbb{R})$. The irreducible representation r_1^e is intertwined with the subrepresentation $\overline{\pi}^-_{1/2}$ of $\rho(-1/2,1/2)$ via the operator

$$\Phi(x) \mapsto r_1(g)\Phi(0);$$

the representation r_1^0 is intertwined with the subrepresentation $\overline{\pi}^-_{3/2}$

$$\Phi(x) = \frac{d}{dx}(r_1(q)\Phi)\Big|_{x=0} .$$

Proof (3) (Sketch). A basis for $L^2(\mathbb{R})$ is provided by the functions $\Phi_m(x) = e^{-\pi x^2} H_n(\sqrt{2\pi}\, x)$ when H_m is the Hermite polynomial of degree m, $m \geq 0$. But by differentiation, $r_1(U)\Phi_m = i(m+1/2)\Phi_m$, i.e., r_1^e has lowest weight $1/2$ (corresponding to Φ_0) and r_1^0 has lowest weight $3/2$ (corresponding to Φ_1). Thus $r_1^e = \overline{\pi}^-_{1/2}$ and $r_1^0 = \overline{\pi}^-_{3/2}$.

Corollary 4.20. Let r_1^* denote the representation of $\overline{SL}_2^*(\mathbb{R})$ induced from $r_1(\tau)$ on $\overline{SL}_2(\mathbb{R})$. Then

$$r_1^* = \overline{\pi}_{1/2} \oplus \overline{\pi}_{3/2} .$$

Note that r_1^* is essentially the representation r_1 of Section 2.5 restricted to $\overline{SL}_2^*(\mathbb{R})$.

Corollary 4.20 associates to the trivial representation of $GL_1(\mathbb{R})$ the representation $\overline{\pi}_{1/2}$ of $\overline{GL}_2(\mathbb{R})$ and to the representa-

tion $\mathrm{sgn}(x)$ of $GL_1(\mathbb{R})$ the representation $\overline{\pi}_{3/2}$. Modulo $(0,\infty)$, this is the sought-after correspondence W^1.

Concluding remark. If $F = \mathbb{C}$, $r_1(\tau)$ is independent of τ. Moreover, $r_1(\tau)$ defines an ordinary representation of $SL_2(\mathbb{C})$. In fact Kubota has shown in [23] that the "even" piece of $r_1(\tau)$ is the complementary series representation of $SL_2(\mathbb{C})$ at $s = 1/2$. Thus this complementary series representation once again appears as the analogue of $\overline{\pi}_{1/2}$. The "odd" piece of $r_1(\tau)$ for $F = \mathbb{C}$ is equivalent to the representation of $SL_2(\mathbb{C})$ induced from the character $z \to z/|z|$ of the subgroup $\{\begin{bmatrix} z & w \\ 0 & z^{-1} \end{bmatrix}\}$. The intertwining operator is constructed just as in Proof (2) of Theorem 4.19. The fact that pieces of r_1 (over arbitrary fields) could be identified in this way was pointed out to Sally and myself by Howe; cf. his Zentralblatt review of [10].

§5. Local Theory: the p-adic places.

Throughout this Section, F will denote a non-archimedean field of characteristic zero. For simplicity, the residual characteristic of F will be assumed to be odd. For dyadic fields, see the remarks after Lemma 5.6. The metaplectic cover of

$$G = GL_2(F)$$

(constructed in Section 2.2) will be denoted by $\overline{G}$.

The purpose of this Section is to describe the local map

$$S: \overline{\pi} \to \pi$$

for certain irreducible unitary representations of $\overline{G}$, namely the non-supercuspidal representations.

5.1. Basic Representation Theory.

Suppose $\overline{\pi}$ is a (not-necessarily preunitary) representation of $\overline{G}$ on a complex vector space V. Then $\overline{\pi}$ is said to be admissible if (i) for every $v \in V$, the stabilizer of v in $\overline{G}$ is an open subgroup of $\overline{G}$, and (ii) for every open compact subgroup $\overline{G}'$ of $\overline{G}$, the space of vectors stabilized by $\overline{G}'$ is finite dimensional.

If $\overline{\pi}$ is irreducible and admissible, we say $\overline{\pi}$ is supercuspidal if for every vector v in V,

$$\int_U \overline{\pi}(u)v\,du = 0 \tag{5.1}$$

for some open compact subgroup U of N. Note that (5.1) is meaningful since $\overline{G}$ splits over N.

Now suppose $\overline{\pi}$ is an irreducible admissible non-supercuspidal representation of $\overline{G}$. Let $V(\overline{\pi},N)$ denote the set of vectors v in V satisfying (5.1) for some U as above. Since N is the union of all its open compact subgroups U, $V(\overline{\pi},N)$ is actually a subspace of V.

Since $\overline{B}$ normalizes V, $\overline{\pi}(\overline{B})$ preserves $V(\overline{\pi},N)$. Thus we obtain in $V/V(\overline{\pi},N)$ a representation $\overline{\sigma}'$ of $\overline{B}$ (actually $\overline{B}/N$,

since $\pi(n)v-v \in V(\pi,N)$ for all $v \in V$ and $n \in N$). Note that $\overline{\sigma}'$ will be non-trivial on Z_2 if and only if $\overline{\pi}$ is genuine on $\overline{G}$.

The significance of our assumption that $\overline{\pi}$ not be supercuspidal is that $V/V(\overline{\pi},N)$ (hence $\overline{\sigma}'$) is then non-zero. Consequently, $\overline{\pi}$ can be imbedded in the G-module induced from an irreducible quotient of $V/V(\overline{\pi},N)$.

To be more precise, fix v a non-zero vector in V. Since $\overline{\pi}$ is irreducible, the vectors $\overline{\pi}(\overline{g})v$, $\overline{g} \in \overline{G}$, span V. If $\overline{G}'$ is an open compact subgroup in the stabilizer of v, and $\{g_i\}$ is a finite set of representatives $\overline{B}\backslash G/\overline{G}'$, transforms under $\overline{B}$ of the finite number of vectors $v_i = \overline{\pi}(\overline{g}_i)v$ also span V. That is, $V/V(\overline{\pi},N)$ is a $\overline{B}$-module of finite type. Thus there is an invariant subspace V' of V which contains $V(\overline{\pi},N)$ and is such that the natural representation $\overline{\sigma}$ of $\overline{B}/N$ in $V/V' = W$ is irreducible.

As always, let $\delta_{\overline{B}}$ denote the modular function for $\overline{B}$. To imbed $\overline{\pi}$ in $\mathrm{Ind}(\overline{B},\overline{G},\overline{\sigma})$, define, for each v in V,

$$f_v(\overline{g}) = \underline{\overline{\pi}(\overline{g})v}$$

where $\underline{v}$ denotes the class of v in $W = V/V'$. Clearly $f_v:\overline{G} \to W$ satisfies

$$f_v(\overline{b}\,\overline{g}) = \delta_{\overline{B}}^{1/2}(\overline{b})\overline{L}(\overline{b})f_v(\overline{g}) \tag{5.2}$$

where $\overline{L} = \delta_{\overline{B}}^{-1/2}\,\overline{\sigma}$. Thus $v \to f_v$ is an intertwining operator between $\overline{\pi}$ and $\mathrm{Ind}(\overline{B},\overline{G},\overline{L})$ (non-zero since $\overline{\pi}$ is not supercuspidal). That is, $\overline{\pi}$ is a subrepresentation of $\mathrm{Ind}(\overline{B},\overline{G},\overline{L})$. (We are interpreting $\mathrm{Ind}(\overline{B},\overline{G},\overline{L})$ as the space of functions $f:\overline{G} \to \overline{W}$ satisfying (5.2) in addition to being right invariant by some open subgroup of $\overline{G}$.)

If $\overline{\pi}$ is trivial on Z_2 this conclusion is a special case of a general theorem of Jacquet and Harish-Chandra classifying the irreducible admissible non-supercuspidal representations of a p-adic

reductive group.

Now let $\overline{L}$ denote an arbitrary irreducible (finite dimensional) representation of $\overline{B}/N$. Henceforth we shall deal exclusively with irreducible unitary representations of $\overline{G}$ which are equivalent to

$$\mathrm{Ind}(\overline{B},\overline{G},\overline{L})$$

or some unitarizable subrepresentation thereof. To analyze such representations, the theorem below is crucial.

Let B_0 denote the subgroup of B whose diagonal entries have even v-adic order (i.e., the diagonal entries are squares modulo units). By Lemma 2.11, B_0 lifts homomorphically as a subgroup of $\overline{B}$. In particular, $\overline{A}_0 = \overline{B}_0/N = A_0 \times Z_2$ is abelian and its irreducible (finite-dimensional) representations are of the form

$$\overline{L}^0(\mu_1,\mu_2)(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix},\zeta) = \zeta\mu_1(a_1)\mu_2(a_2) \tag{5.3}$$

with μ_1,μ_2 quasicharacters of F^x.

Theorem 5.1. Suppose $\overline{L}$ is an irreducible finite dimensional representation of $\overline{A} = \overline{B}/N$ on a complex vector space V. If $\overline{L}$ is trivial on Z_2, V is one-dimensional, and $\overline{L}(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},\zeta) = \mu_1(a)\mu_2(b)$ for some choice of μ_1,μ_2. On the other hand, if $\overline{L}$ is non-trivial on Z_2, V is four dimensional and $\overline{L}$ is equivalent to

$$\overline{L}(\mu_1,\mu_2) = \mathrm{Ind}(\overline{B}_0,\overline{B},\overline{L}^0(\mu_1,\mu_2)$$

with $\overline{L}^0(\mu_1,\mu_2)$ as in (5.3)).

The first part of this Theorem is obvious. To prove the second we shall exploit the theory of representations for groups with normal subgroup of finite index. This theory goes back to Weyl and Clifford and has already been described for unitary representations in Lemma 4.3.

As before, suppose N is a normal subgroup of index two in G. If L_0 is any finite-dimensional irreducible representation of N, let H denote the subgroup of h in G such that $(L_0)^h = L_0$. Clearly $H = G$ if L_0 is self-conjugate and equals N otherwise.

Lemma 5.2. Suppose L^* is any irreducible finite-dimensional representation of $H(L_0)$ whose restriction to N contains L_0. Then

(1) $\mathrm{Ind}(H,G,L^*)$ is irreducible,

(2) the restriction of $\mathrm{Ind}(H,G,L^*)$ is irreducible if L_0 is selfconjugate and equivalent to $L_0 \oplus L_0^h$ otherwise;

(3) all finite-dimensional irreducible representations of G are so obtained.

This lemma is essentially the content of Section 12 of Chapter III of [3]. Thus we shall not include a proof here. (In [3] Boerner assumes that G contains an element of order two outside N but this assumption is not necessary.)

Now consider the normal subgroup B_+ of B consisting of matrices whose diagonal elements are either both squares modulo units or both non-squares modulo units. Clearly $[\overline{A}_0:\overline{A}_+] = [\overline{A}_+:\overline{A}] = 2$. Thus we can prove Theorem 5.1 by applying Lemma 5.2 to these pairs.

From elementary properties of Hilbert's symbol it follows that

$$(5.4)\qquad ([\begin{smallmatrix} a_1' & 0 \\ 0 & a_2' \end{smallmatrix}],\zeta')([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],\zeta)([\begin{smallmatrix} a_1' & 0 \\ 0 & a_2' \end{smallmatrix}],\zeta')^{-1}$$

$$= ([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],\zeta)(L_2,(a_1',a_2)(a_2',a_1))$$

for arbitrary $a_1,a_2,a_1',a_2' \in F^x$. In particular, fixing $h = ([\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}],1)$ in $\overline{A}_+$,

$$\overline{L}^0(\mu_1,\mu_2)^h = \overline{L}_0(\mu_1',\mu_2')$$

with $\mu_i'(x) = (\omega,x)\mu_i(x)$. In other words, if $N = \overline{A}_0$, $G = \overline{A}_+$, and $L_0 = \overline{L}^0(\mu_1,\mu_2)$, then $H(L_0) = \overline{A}_0$. Consequently

$$\overline{L}' = \mathrm{Ind}(\overline{A}_0,\overline{A}_+,\overline{L}^0(\mu_1,\mu_2))$$

is irreducible and all irreducible finite dimensional representations of $\overline{A}_+$ non-trivial on Z_2 are so obtained.

Now apply Lemma 5.2 to $\overline{A}_+, \overline{A}$, and $\overline{L}'$. Fixing $h = ([\begin{smallmatrix}\omega & 0\\ 0 & 1\end{smallmatrix}],1)$ in $\overline{A}$ it follows from (5.4) that $(\overline{L}:)^h$ is not equivalent to $\overline{L}'$. Indeed

$$\overline{L}^0(\mu_1,\mu_2)^h([\begin{smallmatrix}a_1 & 0\\ 0 & a_2\end{smallmatrix}],\zeta) = (\omega,a_2)\overline{L}^0(\mu_1,\mu_2)([\begin{smallmatrix}a_1 & 0\\ 0 & a_2\end{smallmatrix}],\zeta),$$

and the restriction of $\overline{L}'$ to $\overline{A}_0$ is $\overline{L}^0(\mu_1,\mu_2)\oplus \overline{L}^0(\mu_1',\mu_1')$ (with μ_1',μ_2' as above). Hence every irreducible representation of $\overline{A}$ is of the form $\mathrm{Ind}(\overline{A}_+,\overline{A},\overline{L}'(\mu_1,\mu_2)$ and Theorem 5.1 is proved. □

Corollary 5.3. With respect to an appropriate basis, the restriction of $\mathrm{Ind}(\overline{B}_0,\overline{B},\overline{L}^0(\mu_1,\mu_2))$ to $\overline{A}_0$ takes the form

(5.5)

$$\begin{bmatrix} \overline{L}^0(\mu_1,\mu_2)([\begin{smallmatrix}a_1 & 0\\ 0 & a_2\end{smallmatrix}],\zeta) & 0 & 0 & 0 \\ 0 & (\omega,a_1a_2)\overline{L}^0 & 0 & 0 \\ 0 & 0 & (\omega,a_2)\overline{L}^0 & 0 \\ 0 & 0 & 0 & (\omega,a_1)\overline{L}^0 \end{bmatrix}$$

We close this Subsection with some Lemmas which will be useful in our analysis of the representations $\mathrm{Ind}(\overline{B},\overline{G},\overline{L})$ and in our definition of the local map for F.

Lemma 5.4. Suppose $\overline{L}$ is a finite-dimensional genuine representation of $\overline{A}$. Then its character $\chi_{\overline{L}}$ vanishes on elements of $\overline{A}$ whose projections in A are not squares.

Proof. Clearly $\chi_{\overline{L}}$ is a class function satisfying

$$\chi_{\overline{L}}(\zeta\,\overline{a}) = \zeta\,\chi_{\overline{L}}(\overline{a})$$

for all $\overline{a} \in A$ and $\zeta \in Z_2$. On the other hand, if the projection of $\overline{a}$ in A is not a square, by Corollary 2.12

$$\overline{a} \neq \overline{\gamma}\,\overline{a}\,\overline{\gamma}^{-1} = \overline{a}(1,-1)$$

for some $\bar{\gamma} \in \bar{A}$. Thus our claim is immediate. □

Lemma 5.5. The character of $\bar{L}(\mu_1,\mu_2)$ is computable from Corollary 5.3. Indeed suppose $\bar{a} = [\begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix}, \zeta)$. Then

$$\chi_{\bar{L}(\mu_1,\mu_2)}(\bar{a}) = 4\zeta\mu_1(\alpha^2)\mu_2(\beta^2) \tag{5.6}$$

Proof. Hilbert's symbol is trivial on squares. □

Lemma 5.6. Suppose

$$\mu_i(a) = \nu_i(a) \quad (i = 1,2)$$

for all squares $a \in F^x$; equivalently, $(\mu_i)^2 = (\nu_i)^2$ on F^x. Then $\bar{L}(\mu_1,\mu_2)$ is equivalent to $\bar{L}(\nu_1,\nu_2)$; in particular multiplication by characters of order 2 is irrelevant.

Proof. Compare characters. □

Concluding Remarks (the case of a dyadic field).

Suppose the residual characteristic of F is even, say 2^n. Let A_0 denote the subgroup of matrices $[\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}]$ where both a_1 and a_2 have even v-adic order and a_1 has unit part congruent to a square modulo 40. Then $\bar{A}$ splits over A_0, and each of its irreducible representations $\bar{L}$ (non-trivial on Z_2) is induced from a one-dimensional representation (μ_1,μ_2) of $\bar{A}_0$ (tensored with the non-trivial character of Z_2). But $\bar{A}_0$ has index $[F^x: (F^x)^2]$ in $\bar{A}$. Thus $\bar{L}$ has dimension $4(2^n)^{\mathrm{ord}_v(2)}$ (cf. Theorem 5.1 where $[F^x:(F^x)^2] = 4$). Moreover, Lemma 5.6 is still valid. Since a detailed discussion of the representation theory of the metaplectic group over a dyadic field will appear elsewhere, henceforth we shall deal exclusively with the case of odd residual characteristic (except possibly for a few parenthetical remarks).

5.2. Class 1 representations.

In this paragraph we begin a careful analysis of the induced representation

$$\overline{\rho}(\mu_1,\mu_2) = \mathrm{Ind}(\overline{B},\overline{G},\overline{L}(\mu_1,\mu_2)).$$

Denote its space by $\overline{B}(\mu_1,\mu_2)$ and keep in mind that $\overline{\rho}(\mu_1,\mu_2)$ is non-trivial on Z_2.

Lemma 5.7. The restriction of $\overline{\rho}(\mu_1,\mu_2)$ to the dense subspace of $\overline{K}$-finite vectors in $\overline{B}(\mu_1,\mu_2)$ is admissible.

Proof. Let $V(\overline{L})$ denote the four-dimensional space of $\overline{L}(\mu_1,\mu_2)$. Then the dense subspace of $\overline{K}$-finite vectors in $B(\mu_1,\mu_2)$ consists of locally constant $V(\overline{L})$-valued functions on $\overline{G}$ satisfying

$$\varphi(([\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}],\zeta)\overline{g}) = \left|\frac{a_1}{a_2}\right|^{1/2}\overline{L}([\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}],\zeta)\varphi(\overline{g})$$

for all $([\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}],\zeta) \in \overline{B}$. On the other hand, by Iwasawa's decomposition for G,

$$\overline{G} = \overline{A}\ N\ K. \tag{5.6}$$

Thus this space is naturally isomorphic to the space of locally constant $V(L)$-valued functions on K satisfying

$$\varphi([\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}]k) = \overline{L}([\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}],1)\varphi(k)$$

for all $[\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}] \in B \cap K$. (Since a_1 and a_2 are units in F^x, $\left|\frac{a_1}{a_2}\right| = 1$.)

To prove admissibility, fix an open group $\overline{G}'$ of $\overline{G}$ and let $H(\overline{G}')$ denote the subspace of K-finite vectors in $\overline{B}(\mu_1,\mu_2)$ stabilized by $\overline{G}'$. Since each φ in $H(\overline{G}')$ is left fixed by the compact open subgroup $\overline{G}' \cap \overline{K}$, each such φ is determined by its values on the finite set $\overline{K}/\overline{G}' \cap \overline{K}$. Consequently $H(\overline{G}')$ is finite-dimensional, and the non-trivial part of admissibility is proved. □

Definition 5.8. A genuine representation of $\overline{G}$ will be called class 1 of its restriction to $\overline{K}$ contains the representation

$$\overline{\sigma}_0:(k,\zeta) \to \zeta$$

at least once; equivalently, its restriction to K contains the identity representation.

Theorem 5.9. $\overline{\rho}(\mu_1,\mu_2)$ is class 1 if and only if both μ_1^2 and μ_2^2 are unramified. In either case the identity representation occurs at most once.

Proof. Each φ in $B(\mu_1,\mu_2)$ satisfies

$$\varphi([\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}],1) = |\frac{a_1}{a_2}|^{1/2} \overline{L}(\mu_1,\mu_2)([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],1)\varphi(e) \tag{5.7}$$

for all $[\begin{smallmatrix} a_1 & x \\ 0 & a_2 \end{smallmatrix}] \in B$. Moreover, if φ is right K-invariant (hence identically equal to $\varphi(e)$ on K),

$$\overline{L}(\mu_1,\mu_2)([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],1)\varphi(e) = \varphi(e) \tag{5.8}$$

for all $[\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}] \in A \cap K$. Thus $\overline{\rho}(\mu_1,\mu_2)$ is class 1 only if (the space of K-finite vectors in) $B(\mu_1,\mu_2)$ contains some function φ satisfying (5.8). In particular, by (5.8) $\overline{\rho}(\mu_1,\mu_2)$ is class 1 if and only if (5.8) obtains for some such φ.

From the Corollary to the proof of Theorem 5.1, recall that the eigenvalues of $\overline{L}(\mu_1,\mu_2)([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],1)$ are

$$\lambda_1 = \mu_1(a_1)\mu_2(a_2)$$
$$\lambda_2 = (\omega,a_1a_2)\lambda_1$$
$$\lambda_3 = (\omega,a_2)\lambda_1$$

and

$$\lambda_4 = (\omega,a_1)\lambda_1.$$

Thus $\overline{\rho}(\mu_1,\mu_2)$ is class 1 if and only if one of these λ_i is 1

for all $a_1, a_2 \in O^x = U$. But $(\omega, \cdot)$ defines a ramified character of F^x. Hence (5.8) is impossible if either μ_1^2 or μ_2^2 is ramified.

On the other hand, suppose both μ_1^2 and μ_2^2 are unramified. Then by Lemma 5.6 we may assume μ_1 and μ_2 are also unramified. In this case, $\lambda_1 \equiv 1$ and consequently $\rho(\mu_1, \mu_2)$ is class 1.

Note finally that when $\lambda_1 = 1$, λ_2, λ_3, and λ_4 are non-trivial. Indeed these eigenvalues then define ramified characters of F^x. Thus the space of functions in $\overline{B}(\mu_1, \mu_2)$ satisfying (5.8) comprises a one-dimensional space and the proof of the theorem is complete. (This space consists precisely of those φ whose restrictions to K are constant and take their values in the one-dimensional subspace of $V(\overline{L})$ belonging to λ_1.) $\square$

Analogous results hold for dyadic fields with K^N replacing $GL_2(O)$.

5.3. Hecke operators.

Let $H(\overline{G},\overline{\sigma}_0)$ denote the Hecke algebra of continuous compactly supported functions ψ on G satisfying

$$\psi(\overline{k}\ \overline{g}\ \overline{k}') = \overline{\sigma}_0(\overline{k})\psi(\overline{g})\overline{\sigma}_0(\overline{k}') \tag{5.9}$$

for all $\overline{k},\overline{k}'$ in $\overline{K}$. Multiplication in $H(G,\overline{\sigma}_0)$ is given by the convolution product

$$\psi_1*\psi_2(\overline{x}) = \int_{\overline{G}} \psi_1(\overline{x}\ \overline{y}^{-1})\psi_2(\overline{y})d\overline{y}. \tag{5.10}$$

Note that if ω is any continuous function on $\overline{G}$ satisfying $\omega(\overline{g}\zeta) = \zeta\omega(\overline{g})$, the integral in (5.10) is still well-defined. In particular, if $\varphi_0(\overline{g})$ is any zonal spherical function in $\overline{B}(\mu_1,\mu_2)$ (satisfying $\varphi_0(\overline{g}\ \overline{k}) = \overline{\sigma}_0(\overline{k})\varphi_0(\overline{g})$), $\varphi_0*\psi(\overline{g})$ is also zonal spherical, and

$$\varphi_0*\psi(\overline{g}) = \lambda(\psi)\varphi_0(\overline{g})$$

for all $\psi \in H(\overline{G},\sigma_0)$.

Our purpose in this Subsection is to compute $\lambda(\psi)$ for some important special ψ. The required double coset space analysis is contained in the two lemmas below.

Lemma 5.10. Fix n and m in $\mathbb{Z}$. Then

$$\overline{K}\begin{bmatrix}\omega^n & 0\\ 0 & \omega^m\end{bmatrix},1)\overline{K} = \bigcup_{\zeta\in Z_2} K\begin{bmatrix}\omega^n & 0\\ 0 & \omega^m\end{bmatrix},1)K(1,\zeta) \tag{5.11}$$

and the union is disjoint if and only if n and m are even (i.e. ω^n and ω^m are squares).

Proof. Only the second assertion is non-trivial. To prove it, suppose first that $\omega^n = \omega\cdot\omega^{2\ell}$ with $\ell \in \mathbb{Z}$. If we choose a unit ε in $F^\times$ such that $(\omega,\varepsilon) = -1$, then

$$\begin{bmatrix}1 & 0\\ 0 & \varepsilon\end{bmatrix},1)\begin{bmatrix}\omega^n & 0\\ 0 & \omega^n\end{bmatrix},1)(1,-1)$$

$$= \left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, 1\right) .$$

This implies, however, that

$$K\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)K(1,-1) = K\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)K .$$

Thus the union in (5.11) is not disjoint if n (say) is not even.

Suppose now that both ω^n and ω^m are squares. If the two sets on the right side of (5.11) are not disjoint then they coincide. Consequently

$$(5.12) \qquad (k_1,1)\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right) = \left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)(k_2,1)(1,-1)$$

for <u>some</u> k_1, k_2 in K. But now both ω^n and ω^m are squares. Thus $\beta\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}\right),k_2) = \beta(k_1,\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}) = 1$, and (5.12) implies

$$(k_1\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1) = (\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}k_2, -1),$$

an obvious contradiction. □

According to this lemma, a non-trivial element of $H(\overline{G},\overline{\sigma}_0)$ cannot be supported in $\overline{K}\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)\overline{K}$ unless both n and m are even. In particular we have:

<u>Corollary 5.11</u>. The function

$$\psi_{n,m}(\overline{g}) = \begin{cases} \zeta & \text{if } \overline{g} \in K\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)K(1,\zeta) \\ 0 & \text{otherwise} \end{cases}$$

always belongs to $H(\overline{G},\sigma_0)$. However, it is identically zero unless both n and m are even.

Corollary 5.11 explains why Shimura and his predecessors (in treating forms of <u>half</u>-integral weight) dealt mostly with Hecke operators $T_k(\ell)$ corresponding to <u>square</u> ℓ. (Cf. Proposition 1.0 and p.450 of [38]; Shimura's $T(p^2)$ is essentially $\psi_{0,2}$ with $F = \mathbb{Q}$.) Lemma 5.10 appears as Proposition 7 in Kubota [22]. For

even v, and $K = K^N$, $\psi_{1,0}$ is not necessarily zero; again, compare p.450 of [38].

The collection $\{\psi_{n,m}\}$, with n and m even, generates the algebra $H(\overline{G},\overline{\sigma}_0)$ of $\overline{\sigma}_0$-spherical functions. Indeed by the theory of elementary divisors for G,

$$\overline{G} = \bigcup_{n,m} \overline{K}\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)\overline{K} . \tag{5.13}$$

On the other hand, each ψ in $H(\overline{G},\overline{\sigma}_0)$ is compactly supported. Thus by (5.9), ψ is completely determined by its value at a finite number of matrices $\left(\begin{bmatrix} \omega^n & 0 \\ 0 & \omega^m \end{bmatrix}, 1\right)$. That is, each such ψ is a finite linear combination of the functions $\psi_{n,m}$ of Corollary 5.11.

Using a modification of the usual arguments, one can directly show $\psi_{n,m}{}^*\psi_{n',m'} = \psi_{n',m'}{}^*\psi_{n,m}$, hence that $H(\overline{G},\overline{\sigma}_0)$ is abelian. (Cf. Section 4 of [21].) Our interest, however, is in computing eigenvalues for the zonal spherical functions in $\overline{B}(\mu_1,\mu_2)$.

Lemma 5.12. The double coset

$$H_\omega = K\begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}K$$

is the disjoint union of q^2+q left K-cosets. Representatives for these cosets are

$$\begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}\begin{bmatrix} z & 1 \\ -1 & 0 \end{bmatrix} \quad (\text{with } z \in (0/\omega^2 0)^x)$$

and

$$\begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \omega y & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

with $y = 0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-1} \in 0/\omega 0$ and ε a generator of the (q-1)st roots of unity in F^x.

This lemma concerns only G and is classical. Thus we do not include a proof. We do, however, include the following remarks.

In $\overline{G}$,

$$([\begin{smallmatrix} c^{-1} & 0 \\ 0 & \omega^{2-k} c \end{smallmatrix}], 1)([\begin{smallmatrix} \omega^k c & 1 \\ -\omega^2 & 0 \end{smallmatrix}], 1) = ([\begin{smallmatrix} \omega^k & c^{-1} \\ 0 & \omega^{2-k} \end{smallmatrix}], (c,\omega)^k)$$

for each unit c in F. (This follows from elementary computations involving Hilbert's symbol.) Thus if $z = \omega^k c$,

$$(K[\begin{smallmatrix} 1 & 0 \\ 0 & \omega^2 \end{smallmatrix}][\begin{smallmatrix} z & 1 \\ -1 & 0 \end{smallmatrix}], 1)$$

$$= K([\begin{smallmatrix} \omega^k & c^{-1} \\ 0 & \omega^{2-k} \end{smallmatrix}], (c,\omega)^k).$$

On the other hand,

$$(K[\begin{smallmatrix} \omega^2 & 0 \\ 0 & 1 \end{smallmatrix}][\begin{smallmatrix} 1 & 0 \\ \omega y & 1 \end{smallmatrix}][\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}], 1)$$

$$= (K[\begin{smallmatrix} 1 & 0 \\ 0 & \omega^2 \end{smallmatrix}][\begin{smallmatrix} 1 & -\omega y \\ 0 & 1 \end{smallmatrix}], 1) .$$

Theorem 5.13. Suppose $\overline{\rho}(\mu_1,\mu_2)$ is class one and $\varphi_0 \in \overline{B}(\mu_1,\mu_2)$ transforms under $\overline{K}$ according to $\overline{\sigma}_0$. Let $\overline{T}(\omega^2)$ denote the "Hecke operator" corresponding to convolution by $\psi_{0,2}$, and define $\lambda(\omega)$ by

$$\overline{T}(\omega^2)\varphi_0(\overline{x}) = \lambda(\omega)\varphi_0(\overline{x}) .$$

Then if $\mu_i(x) = |x|^{s_i}$,

$$\lambda(\omega) = q(q^{2s_1} + q^{2s_2}) .$$

Proof. By Lemma 5.6, there is no loss of generality in assuming μ_i unramified. Also, by our hypothesis on φ_0,

$$\varphi_0((k,1)) \equiv \overline{v}_0$$

where $\overline{v}_0$ is some distinguished vector in $V(\overline{L})$. (More precisely, $\overline{v}_0$ belongs to the subspace of fixed vectors for $\overline{L}(\mu_1,\mu_2)$ restricted to $A \cap K$.) Thus

$$\lambda(\omega)\overline{v}_0 = \varphi_0 * \psi_{0,2}(e) = \int_{Z_2 \setminus \overline{G}} \varphi_0(y^{-1})\psi_{0,2}(y)\,dy$$

$$= \int_{H_\omega} \varphi_0(y^{-1})\,dy \quad .$$

Now we apply Lemma 5.12. According to the remarks following this Lemma,

$$\lambda(\omega)v_0 = \int_K \varphi_0((k\begin{bmatrix}\omega^2 & 0\\ 0 & 1\end{bmatrix}k)^{-1})\,dk$$

$$+ \sum_{z=\omega^k c} \int_K \varphi_0(\left(\begin{bmatrix}\omega^{-k} & -c^{-1}\\ 0 & \omega^{k-2}\end{bmatrix}, ((\omega c,\omega)^k)(k,1)\right)dk$$

$$+ \sum \int_K \varphi_0(\left(\begin{bmatrix}1 & \omega y\\ 0 & \omega^{-2}\end{bmatrix}, 1\right)(k,1)\,dk$$

But φ_0 belongs to $\overline{B}(\mu_1,\mu_2)$. Thus we have

$$\lambda(\omega)\overline{v}_0 = |\omega^{-1}|\overline{L}(\mu_1,\mu_2)\left(\begin{bmatrix}\omega^{-2} & 0\\ 0 & 1\end{bmatrix}, 1\right)\overline{v}_0$$

$$+ \sum_{\substack{0\neq z=\omega^k c \\ (\mathrm{mod}\,\omega^2 0)}} (c\omega,\omega)^k |\omega^{-k+1}|\overline{L}\left(\begin{bmatrix}\omega^{-k} & -c^{-1}\\ 0 & \omega^{k-2}\end{bmatrix}, 1\right)\overline{v}_0$$

$$+ |\omega| \sum_{y\in 0/\omega 0} L\left(\begin{bmatrix}1 & \omega y\\ 0 & \omega^{-2}\end{bmatrix}, 1\right)\overline{v}_0$$

To continue, we need to further analyze the second sum appearing above.

First consider the set of non-trivial coset representatives $\{z\}$ of 0 modulo $\omega^2 0$. This set has q^2-1 elements and contains $q(q-1)$ units. The remaining $q-1$ representatives are of the form $\omega^k c$ with $k = 1$.

Now consider the sum

$$\sum_{\substack{z=\omega c \\ (\mathrm{mod}\,\omega^2 0)}} (c,\omega)\overline{L}\left(\begin{bmatrix}\omega^{-1} & -c^{-1}\\ 0 & \omega^{-1}\end{bmatrix}, 1\right)\overline{v}_0 \quad .$$

Since $(q-1)$ is even, and $(\epsilon^\ell,\omega) = (-1)^\ell$, this sum actually

vanishes. $(\overline{L}([\begin{smallmatrix}\omega^{-1} & -c^{-1}\\ 0 & \omega^{-1}\end{smallmatrix}],1)\overline{v}_0$ does not depend on c.) Thus the second sum appearing in (*) simply equals $q(q-1)|\omega^{1}|\mu_2(\omega^{-2})\overline{v}_0$ and

$$\lambda(\omega)\overline{v}_0 = q\mu_1(\omega^{-2})\overline{v}_0 + (q-1)\mu_2(\omega^{-2})\overline{v}_0 + \mu_2(\omega^{-2})\overline{v}_0 .$$

That is,

$$\lambda(\omega) = q(q^{2s_1} + q^{2s_2}),$$

as desired. □

In Subsection 5.4 a local map

$$S_v: \overline{\pi} \to \pi$$

will be defined for irreducible quotients of $\overline{\rho}(\mu_1,\mu_2)$ and Theorem 5.13 will be used to explain how this map is consistent with Shimura's correspondence. In the meantime, recall the analogue of Theorem 5.13 for $G = GL_2(F)$.

If $\rho(\mu_1,\mu_2)$ denotes the representation of G induced from the character

$$[\begin{smallmatrix}a_1 & x\\ 0 & a_2\end{smallmatrix}] \to \mu_1(a_1)\mu_2(a_2)$$

of B, $\rho(\mu_1,\mu_2)$ is class 1 iff <u>both</u> μ_1 and μ_2 are unramified. Moreover, if $\mu_i(x) = |x|^{s_i}$, and φ_0 is the (essentially unique) K-fixed function in $B(\mu_1,\mu_2)$, then

$$\int_{K[\begin{smallmatrix}1 & 0\\ 0 & \omega\end{smallmatrix}]K} \varphi_0(hy^{-1})dy = q^{1/2}(q^{s_i} + q^{s_2})\varphi_0(y) \tag{5.12}$$

Note that the operator on the left side of (5.12) is (essentially) Hecke's classical $T(p)$. For details, see [7].

In Section 6 we shall also see that Theorem 5.13 is consistent with Kubota [21], p.53. Indeed our $\overline{T}(\omega^2)$ is Kubota's $(q^2+q)\rho(1,\omega^2)$.

5.4. The Local Map

In this Subsection we shall define the local map $S: \overline{\pi} \to \pi$ for irreducible representations of $\overline{G}$ imbeddable in $\overline{B}(\mu_1,\mu_2)$. We begin by presenting a useful (if not unexpected) character computation.

Let $H(\overline{G})$ denote the algebra of all locally constant compactly supported functions on $\overline{G}$ satisfying $f(\zeta\overline{g}) = \zeta f(\overline{g})$. The product in $H(\overline{G})$ is given by (5.10). If $\overline{\pi}$ is any admissible genuine representation of $\overline{G}$,

$$f \to \overline{\pi}(f) = \int_{Z_2\backslash\overline{G}} f(\overline{g})\overline{\pi}(\overline{g})\,d\overline{g}$$

defines an "admissible" representation of $H(\overline{G})$ on the space of $\overline{\pi}$. In particular, $\overline{\pi}(f)$ has finite rank.

Theorem 5.14. If $f \in H(\overline{G})$, then

$$\operatorname{trace}(\overline{\rho}(\mu_1,\mu_2)(f)) = \int_{Z_2\backslash\overline{G}} f(\overline{g})\overline{\chi}(\mu_1,\mu_2)(\overline{g})\,d\overline{g}$$

where $\overline{\chi}(\mu_1,\mu_2)$ denotes the character of $\overline{\rho}(\mu_1,\mu_2)$: If $\overline{g}$ is conjugate in $\overline{G}$ to some

$$\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},\zeta\right),$$

with both a_1, a_2 squares in $F^\times$, then

$$\overline{\chi}(\mu_1,\mu_2)(\overline{g}) = 4\zeta\{\mu_1(a_1)\mu_2(a_2) + \mu_2(a_1)\mu_1(a_2)\}\left|\frac{a_1a_2}{(a_1-a_2)^2}\right|^{1/2};$$

otherwise, $\overline{\chi}(\overline{g}) \equiv 0$.

Proof. Abusing notation, we shall confuse $\overline{B}(\mu_1,\mu_2)$ with the subspace of functions which are right invariant for some open subgroup of $\overline{G}$. Then, as already remarked, $\overline{\rho}(\mu_1,\mu_2)$ is of finite rank on $B(\mu_1,\mu_2)$. (This follows from the admissibility of $\overline{\rho}(\mu_1,\mu_2)$ and the fact that f has compact support.) Hence the

trace of $\overline{\rho}(\mu_1,\mu_2)$ certainly exists.

Now if $\varphi \in \overline{B}(\mu_1,\mu_2)$,

$$(\overline{\rho}(f)\varphi)(k) = \int_{Z_2\backslash\overline{G}} f(\overline{g})\overline{\rho}(\overline{g})(k)d\overline{g}$$

$$= \int_{Z_2\backslash\overline{G}} \varphi(k\overline{g})f(\overline{g})d\overline{g} = \int_{Z_2\backslash\overline{G}} \varphi(\overline{g})f(k^{-1}\overline{g})d\overline{g}$$

$$= \int_K \int_{Z_2\backslash\overline{B}} \varphi(\overline{b}k_1)f(k^{-1}\overline{b}k_1)d_\ell\overline{b}\, dk_1$$

$$= \int_K \Big(\int_{Z_2\backslash\overline{B}} \delta_{\overline{B}}(\overline{b})^{1/2}\overline{L}(\overline{b})f(k^{-1}\overline{b}k_1)d_\ell\overline{b}\Big)\varphi(k_1)dk_1 \ ,$$

where $\delta_{\overline{B}}([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],\zeta) = |\frac{a_1}{a_2}|$, $\overline{L} = \overline{L}(\mu_1,\mu_2)$, and $d_\ell\overline{b}$ is a left Haar measure on $\overline{B}$. Thus

$$(\overline{\rho}(f)\varphi)(k) = \int_K \mathbb{K}_f(k,k_1)\varphi(k_1)dk_1$$

where

$$\mathbb{K}_f(k,k_1) = \int_{Z_2\backslash\overline{B}} \delta_{\overline{B}}(\overline{b})^{1/2}\overline{L}(\overline{b})f(k^{-1}\overline{b}k_1)d_\ell\overline{b} \ .$$

One can check now that the integral operator $\mathbb{K}$, acting on the space of all locally constant functions from K to $V(\overline{L})$, has range in $\overline{B}(\mu_1,\mu_2)$. Therefore, the trace of $\overline{\rho}(f)$ on $\overline{B}(\mu_1,\mu_2)$ is precisely the trace of $\mathbb{K}$, i.e.

$$\mathrm{tr}\overline{\rho}(f) = \int_K \mathrm{tr}(\mathbb{K}(k,k))dk$$

$$= \int_{Z_2\backslash\overline{A}} \delta_{\overline{B}}(\overline{a})^{1/2}\mathrm{tr}(\overline{L}(\overline{a}))\Big(\int_K \int_N f(k^{-1}\overline{a}nk)dkdn\Big)d\overline{a} \ .$$

To continue, observe that

$$([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],\zeta)([\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}],1) = ([\begin{smallmatrix} 1 & \frac{-a_1 n}{a_1-a_2} \\ 0 & 1 \end{smallmatrix}],1)([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],\zeta)([\begin{smallmatrix} 1 & \frac{a_1 n}{a_1-a_2} \\ 0 & 1 \end{smallmatrix}],1) \ .$$

Thus, making the change of variable $n \to (\frac{a_1-a_2}{a_1})n$ yields the formula

$$\mathrm{tr}\bar{\rho}(f) = \int_{Z_2\backslash\bar{A}} \mathrm{tr}(\bar{L}(\bar{a}))\left|\frac{(a_1-a_2)^2}{a_1a_2}\right|^{1/2}\left(\int_K\int_N f(k^{-1}n^{-1}\bar{a}\,nk)\,dk\,dn\right)d\bar{a}\ .$$

To show that this actually equals

$$\int_{Z_2\backslash\bar{G}} f(\bar{g})\bar{\chi}(\mu_1,\mu_2)(\bar{g})\,d\bar{g}\ ,$$

we shall compute this last expression directly.

Recall the well-known integration formula

$$\int_G h(g)\,dg = \frac{1}{2}\sum_{\{A\}}\int_A \Delta(a)\int_{A\backslash G} h(x^{-1}ax)\,dx\ .$$

Here the summation extends over all equivalence classes of Cartan subgroups A of G, and

$$\Delta(a) = \left|\frac{(a_1-a_2)^2}{a_1a_2}\right|\ .$$

Applying this formula (keeping in mind that the central function $\bar{\chi}(\mu_1,\mu_2)$ vanishes off A^2), we have

$$\int_{Z_2\backslash\bar{G}} f(\bar{g})\bar{\chi}(\mu_1,\mu_2)(\bar{g})\,d\bar{g} = \int_G f(g,1)\bar{\chi}(\mu_1,\mu_2)(g,1)\,dg$$

$$= \frac{1}{2}\int_{A^2}\Delta(a)\bar{\chi}(\mu_1,\mu_2)((a,1))\left(\int_{A\backslash G} f(g^{-1}ag,1)\,dg\right)da\ .$$

However, since $\beta(n,k) = \beta(k,n) = 1$, this last expression equals

$$\frac{1}{2}\int_{A^2}\Delta(a)\bar{\chi}(\mu_1,\mu_2)(a,1)\left(\int_N\int_K f(k^{-1}n^{-1}(a,1)nk)\,dn\,dk\right)da$$

$$= 2\int_{A^2}\left|\frac{(a_1-a_2)^2}{a_1a_2}\right|^{1/2}\{\mu_1(a_1)\mu_2(a_2)$$

$$+ \mu_2(a_1)\mu_1(a_2)\}\left(\int_N\int_K f(k^{-1}n^{-1}(a,1)nk)\,dn\,dk\right)da$$

$$= 4\int_{A^2}\left|\frac{(a_1-a_2)^2}{a_1a_2}\right|^{1/2}\mu_1(a_1)\mu_2(a_2)\left(\int_N\int_K f(k^{-1}n^{-1}(a,1)nk)\,dn\,dk\right)da$$

T complete the proof, recall from Lemma 5.5 that $\mu_1(a_1)\mu_2(a_2) = \frac{1}{4}\,\mathrm{tr}(L([\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}],1))$ when a_1, a_2 are squares in F^x. Thus

$$\int_{Z_2\backslash G} f(\bar{g})\overline{\chi}(\mu_1,\mu_2)(\bar{g})\,d\bar{g} =$$

$$\int_{Z_2\backslash A} \mathrm{tr}(\overline{L}(\bar{a}))\left|\frac{(a_1-a_2)^2}{a_1a_2}\right|^{1/2}\left(\int_K\int_N f(k^{-1}n^{-1}\bar{a}\,nk)\,dn\,dk\right)d\bar{a}$$

$$= \mathrm{tr}\,\overline{\rho}(f),$$

as claimed. □

Theorems 5.13 and 5.14 suggest that we define a map $\overline{\pi} \to \pi$ as follows. If $\overline{\pi}$ is equivalent to $\overline{\rho}(\mu_1,\mu_2)$, set $\pi = S(\overline{\pi})$ equal to $\rho(\mu_1^2,\mu_2^2)$. The problem with this simple definition is that $\rho(\mu_1^2,\mu_2^2)$ and/or $\overline{\rho}(\mu_1,\mu_2)$ may be reducible.

In [18], Jacquet-Langlands establish the following results:

(a) If $\mu_1\mu_2^{-1}(x) \neq |x|$ or $|x|^{-1}$, $\rho(\mu_1,\mu_2)$ is irreducible;

(b) If $\mu_1\mu_2^{-1}(x) = |x|^{-1}$, $\rho(\mu_1,\mu_2)$ contains just one subrepresentation and it is one-dimensional;

(c) If $\mu_1\mu_2^{-1}(x) = |x|$, $\rho(\mu_1,\mu_2)$ again contains <u>one</u> subrepresentation, this time infinite dimensional and "special".

If $\rho(\mu_1,\mu_2)$ <u>is</u> irreducible, it will be denoted $\pi(\mu_1,\mu_2)$; if not, $\pi(\mu_1,\mu_2)$ denotes <u>the</u> irreducible subrepresentation of $\rho(\mu_1,\mu_2)$. In either case, the obvious equivalences obtain.

(d) If $\mu_1\mu_2^{-1}(x) \neq |x|$ or $|x|^{-1}$, $\pi(\mu_1,\mu_2)$ is <u>unitarizable</u> if:

(i) both μ_1 and μ_2 are unitary (in which case $\pi(\mu_1,\mu_2)$ is a <u>continuous series representation</u>) <u>or</u>:

(ii) $\mu_1\mu_2^{-1}(x) = |x|^{\sigma}$, $\mu_2(x) = \overline{\mu_1(x)}^{-1}$, and $0 < \sigma < 1$ (in which case $\pi(\mu_1,\mu_2)$ is called a <u>complementary series representation</u>).

If $\mu_1\mu_2^{-1}(x) = |x|$ or $|x|^{-1}$, $\pi(\mu_1,\mu_2)$ is unitarizable iff

its central character is already unitary; in this case, $\pi(\mu_1,\mu_2)$ is called a special representation (if it is not one-dimensional).

Using [10] one should be able to obtain analogous results for $\overline{G}$:

(a)' The representation $\overline{\rho}(\mu_1,\mu_2)$ is irreducible if $\mu_1\mu_2^{-1}(x) \neq |x|^{1/2}$ or $|x|^{-1/2}$; in this case, $\overline{\rho}(\mu_1,\mu_2)$ is denoted by $\overline{\pi}(\mu_1,\mu_2)$;

(b)' If $\mu_1\mu_2^{-1}(x) = |x|^{-1/2}$, $\overline{\rho}(\mu_1,\mu_2)$ contains one subrepresentation, which we still denote by $\overline{\pi}(\mu_1,\mu_2)$; this representation is always infinite-dimensional and sometimes class 1 (namely when μ_1^2 and μ_2^2 are unramified and v is odd);

(c)' If $\mu_1\mu_2^{-1}(x) = |x|^{1/2}$, $\overline{\rho}(\mu_1,\mu_2)$ again contains just one subrepresentation, a special representation $\overline{\pi}(\mu_1,\mu_2)$; this representation is infinite-dimensional but never class 1;

(d)' The representations in (a)' are unitarizable if either:

(i) μ_1 and μ_2 are already unitary (this gives the continuous series); or

(ii) $\mu_2(x) = \overline{\mu_1(x)}^{-1}$, and $\mu_1\mu_2^{-1}(x) = x^{\sigma}$, with $0 < \sigma < 1/2$ (this gives the complementary series). The representations in (b)' and (c)' are unitarizable if and only if $\mu_1\mu_2$ is unitary. Moreover, the obvious equivalences obtain.

Now we make the following definition.

Definition 5.15. The map

$$S:\overline{\pi} \to \pi$$

is defined for irreducible unitary representations $\overline{\pi} = \overline{\pi}(\mu_1,\mu_2)$ by setting $S(\overline{\pi})$ equal to $\pi(\mu_1^2,\mu_2^2)$.

Proposition 5.16. The map $\overline{\pi} \to \pi$ is one-to-one and takes class one representations to class one representations.

Proof. If $\pi(\mu_1^2,\mu_2^2) = \pi(\nu_1^2,\nu_2^2)$, then by (c) above $(\mu_1^2,\mu_2^2) = (\nu_1^2,\nu_2^2)$ or (ν_2^2,ν_1^2). In either case, $\overline{\pi}(\mu_1,\mu_2) = \overline{\pi}(\nu_1,\nu_2)$ by Theorem 5.14. Thus S is one-to-one. On the other hand, $\pi(\mu_1^2,\mu_2^2)$ is class 1 iff μ_1^2 and μ_2^2 are unramified, so $\overline{\pi}(\mu_1,\mu_2)$ is class 1 iff $S(\overline{\pi})$ is (Theorem 5.13). □

Proposition 5.17. The map S is natural from several points of view, namely:

(a) S takes special representations to special representations and "trivial" representations to "trivial" representations,

(b) S preserves eigenvalues for the Hecke operators;

(c) Roughly speaking, S preserves characters.

Proof. A "trivial" representation for $GL_2(F)$ (resp. $\overline{GL}_2(F)$) is a one-dimensional representation (resp. a $\overline{\pi}(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1} = |x|^{-1/2}$). Thus Part (a) follows from (b), (c), (b)' and (c)' above.

To prove (b), suppose (for simplicity) that the global field is $\mathbb{Q}$ so $F_v = \mathbb{Q}_p$. Let $\overline{T}(p^2)$ (resp. $T(p)$) denote convolution by $\psi_{0,2}$ (resp. the characteristic function of $K_p\left[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right]K_p$). Now, as always, fix k to be an odd integer. Let $T_{k/2}(p^2)$ (resp. $T_{k-1}(p)$) denote the operator $p^{k/2-2}\,\overline{T}(p^2)$ (resp. $p^{\frac{k-1}{2}-1}\,T(p)$); cf. Shimura [38], p.450. Finally, let $S(\varphi_0)$ denote "the" K-invariant function for $B(\mu_1^2,\mu_2^2)$ when both μ_1^2 and μ_2^2 are unramified (i.e. $\mu_1^2(x) = |x|^{2s_i}$). Then by Theorem 5.13, and the remarks immediately following it, the eigenvalues for φ_0 and $S(\varphi_0)$ with respect to $T_{k/2}(p^2)$ and $T_{k-1}(p)$ are identical.

To explain (c), let $\chi(\mu_1,\mu_2)$ denote the character of the representation $\rho(\mu_1,\mu_2)$ of G. If $\gamma \in A$ is such that γ^2 is regular, put $\Delta_+(\gamma)$ equal to $|(\gamma_1+\gamma_2)^2/\gamma_1\gamma_2|^{1/2}$. Then by Theorem 5.14 and its well-known analogue for G,

$$\chi(\mu_1,\mu_2)(\gamma) = \Delta_+(\gamma)\overline{\chi}(\mu_1,\mu_2)(\iota(\gamma^2)) \quad .$$

(Recall $\iota(\gamma)$ denotes the lift $\gamma \to (\gamma,1)$.) □

Remark 5.18. A crucial feature of Shimura's map from $S_{k/2}(N,\chi)$ to $S_{k-1}(N_0,\chi^2)$ is that it preserves eigenvalues for the Hecke ring. Our map is consistent with Shimura's in this respect as well.

5.5 The Basic Weil Representation.

As always, F is a non-archimedean local field with odd residual characteristic and τ is the canonical non-trivial additive character of F whose conductor is O_F. Fix $q = q_1$. As in Subsection 4.4, we shall not deal directly with the Weil representation r_{q_1}. Rather we shall deal with $r_1(\tau)$. This is the representation of $\widetilde{SL}_2(F)$ in $L^2(F)$ given by the formulas

$$r_1(\tau)\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right)\Phi(x) = \tau(bx^2)\Phi(x) \tag{5.13}$$

$$r_1(\tau)(\overline{w})\Phi(x) = \gamma(q_1,\tau)\hat{\Phi}(x) \tag{5.14}$$

and

$$r_1(\tau)\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right)\Phi(x) = (a,a)|a|^{1/2}\frac{\gamma(q_1,\tau_a)}{\gamma(q_1,\tau)}\Phi(ax)\ . \tag{5.15}$$

Note that $r_1(\tau)\Phi$ is even (resp. odd) if Φ is even (resp. odd).

Theorem 5.19. Let r_e denote the restriction of $r_1(\tau)$ to the space of even functions in $L^2(F)$ and r_0 the restriction to odd functions. Then:

(a) $r_1(\tau) = r_e \oplus r_0$;

(b) r_e and r_0 are irreducible;

(c) (R. Howe) r_0 is supercuspidal;

(d) r_e is equivalent to an irreducible quotient of the class one non-unitary principal series representation $\rho(1/2,1)$ of $\widetilde{SL}_2(F)$.

Proof (Sketch).

(a) Obvious.

(b) Formulas (5.13)-(5.15) imply that the only operators commuting with $r_1(\tau)$ are the identity operator and the reflection operator

$$\Phi(x) \to \Phi(-x).$$

But this latter operator acts "trivially" in the space of even or odd functions in $L^2(F)$. Hence (b) is immediate.

(c) To prove r_0 is supercuspidal it suffices to prove that for all vectors v in the space of r_0,

$$\int_N r_0(u)v\,du = 0,$$

i.e., for all odd Φ in $L^2(F)$ and $x \in F$,

$$\Phi(x) \int_{O_F} \tau(nx^2)\,dn = 0 . \tag{5.16}$$

If $x = 0$ this is obvious. Indeed $\Phi(x) = 0$ by the oddness of Φ.

So suppose $x \neq 0$. Recall from Example 2.29 that $r_1(\tau)$ is equivalent to $r_1(\tau_{t^2})$ for any $t \in F^\times$. This implies we may assume x^2 lies outside O_F, i.e., $\tau(nx^2)$ is non-trivial on O_F. Thus

$$\int_{O_F} \tau(nx^2)\,dn = 0$$

and the proof of (c) is complete.

(d) Let $\overline{B}_1$ denote the inverse image in $\overline{SL}_2(F)$ of the upper triangular subgroup $\{[\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}]\}$ of $SL_2(F)$. By definition, $\rho(1/2,1)$ is the genuine representation of $\overline{SL}_2(F)$ induced from the character

$$\{[\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}],\zeta\} \mapsto \zeta|a|^{1/2}[(\tau,-x)(\tau,\tau)^{1/2}][1-(\mathfrak{c},x)]/2$$

of $\overline{B}_1$. For more details, see [10].

Since the residual characteristic of F is odd, both r_e and $\rho(1/2,1)$ are class 1 with respect to $K = SL_2(O_F)$. Thus to prove (d) it suffices to compare eigenvalues for the corresponding zonal spherical functions. For r_e this is a straightforward p-adic computation. Indeed the K-fixed vector is simply the characteristic function of O_F. For $\rho(1/2,1)$, the computations are similar to those made in the proof of Theorem 5.13. □

Alternate proof for (c). An irreducible admissible representation of $\widetilde{SL}_2(F)$ in some vector space V is super-cuspidal iff no non-trivial N-morphism exists between V and the trivial N-space $\mathbb{C}$. For r_1 acting in the Schwartz-Bruhat space $\mathcal{S}(F)$ the action of $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$ is given by multiplication by the quadratic character $\chi(nq(x))$, i.e., the only N-invariant functional on V is a distribution on F supported at the origin. But the only such distribution (for non-archimedean F) is $\Phi(x) \to \Phi(0)$. Since this is trivial for odd Φ, the proof is complete.

Alternate proof for (d). Following (c), use (5.15) to check that $\Phi(x) \to r(g)\Phi(0)$ intertwines r_1^e with the desired principal series representation.

Concluding remarks. The obvious analogues of (a) - (d) hold for dyadic fields too.

§6. Global theory and odds and ends

Throughout this Section F will denote an arbitrary number field. By $L^2(\overline{X})$ we denote the space of square-integrable functions on $\overline{X} = Z_\infty^0 G_F \backslash G_{\mathbb{A}}$ which are genuine, i.e. such that $\varphi(\overline{g}\zeta) = \zeta\varphi(\overline{g})$ for all $\overline{g} \in \overline{G}_{\mathbb{A}}$ and $\zeta \in Z_2$. The regular representation of $\overline{G}_{\mathbb{A}}$ in $L^2(\overline{X})$ is denoted by $\overline{T}$. Also $L_0^2(\overline{X})$ denotes the subspace of $L^2(\overline{X})$ consisting of cuspidal functions and $L_0^2(\overline{E})$ denotes the subspace generated by poles of Eisenstein series on $\overline{G}_{\mathbb{A}}$.

Let $L_c^2(\overline{X})$ denote the direct integral with respect to Lebesgue measure of the spaces of genuine principal series representations $R(\overline{x}, s)$ with $\mathrm{Re}(s) = 0$ and $\mathrm{Im}(s) > 0$. According to Section 3, $L_c^2(\overline{X})$ is isomorphic to a subspace of $L^2(\overline{X})$ whose orthocomplement decomposes discretely. In fact

$$\overline{T} = \overline{T}_0 \oplus \overline{T}_E \oplus \overline{T}_c$$

where

(i) $\overline{T}_0$ denotes the restriction of $\overline{T}$ to $L_0^2(\overline{X})$;

(ii) $\overline{T}_E$ denotes the restriction of $\overline{T}$ to $L_0^2(\overline{E})$; and

(iii) $\overline{T}_c$, the restriction of $\overline{T}$ to $L_c^2(\overline{X})$, exhausts the continuous spectrum of $\overline{T}$.

Since the decomposition of $\overline{T}$ is known, it remains to describe $\overline{T}_0 \oplus \overline{T}_E$, the discrete spectrum of $\overline{T}$.

6.1. The discrete non-cuspidal spectrum

For convenience, replace $\overline{X}$ by

$$\overline{X}^* = Z_{\mathbb{A}}^2 \, G_F \backslash \overline{G}_{\mathbb{A}} \quad .$$

The corresponding decomposition of the regular representation of $\overline{G}_{\mathbb{A}}$ in $L^2(\overline{X}^*)$ is

$$\overline{T}^* = \overline{T}_0^* \oplus \overline{T}_E^* \oplus \overline{T}_c^* \quad .$$

Here $\overline{T}_E^*$ denotes the discrete non-cuspidal spectrum of $\overline{X}^*$.

Fix $\tau = \prod_v \tau_v$ be any non-trivial character of $F \backslash \mathbb{A}$. For each place v of F let $r_v(q_1, \tau_v)$ denote the corresponding Weil representation of $\overline{SL}_2(F_v)$ in $L^2(F_v)$. Let $r_v^e(q_1, \tau_v)$ denote the restriction of $r_v(q_1, \tau_v)$ to the space of even functions in $L^2(F_v)$. Since τ_v is trivial on O_v for almost every v, $r_v^e(q_1, \tau_v)$ is class 1 for almost every v.

From (2.45) it follows that $r_v^e(q_1, \tau_v)$ is trivial on $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ if -1 is a square in F^x. Then inducing $r_v^e(q_1, \tau_v)$ to $Z_v^2 \backslash \overline{G}_v$ produces a genuine representation of $\overline{G}_v$ which is independent of τ_v. Thus we can assume τ_v is the canonical character of F_v fixed in Subsections 4.4 and 5.5.

The induced representation above, denoted simply $r_v^e(q_1)$, is class 1 for all odd v. Thus the tensor product

$$r_e = \otimes \, r_v^e(q_1)$$

makes sense. In particular, r_e defines a genuine representation of $\overline{G}_{\mathbb{A}}$ trivial on $Z_{\mathbb{A}}^2$.

Theorem 6.1 (cf. [9], [10]). The representation $\overline{T}_E^*$ is equivalent to r_e. In particular, every square integrable automorphic form of half-integral weight which is not a cusp form is a translate of the basic theta function $\theta(z)$ in the sense of Subsection 2.6.

Proof (Sketch). The assertion to be proved is essentially local. Indeed, suppose $\overline{\pi} = \otimes\, \overline{\pi}_v$ is a constituent of $\overline{T}_E^*$. By the theory of Eisenstein series sketched in Section 3, each $\overline{\pi}_v$ is a quotient of some non-unitary principal series representation $\overline{\rho}_v(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1}(x) = |x|^{1/2}$ and $\mu_1^2\mu_2^2 \equiv 1$. But the local results of Subsections 4.4 and 5.5 imply that such $\overline{\pi}_v$ occur in $r_v^e(q_1)$. So since $r_v^e(q_1)$ is easily shown to be irreducible, the Theorem follows. □

The structure of $\overline{T}_E$ (not just $\overline{T}_E^*$) is explained by pasting together the local maps S_v introduced in Subsections 4.2 and 5.4 as follows. Suppose $\overline{\pi} = \otimes\, \overline{\pi}_v$ is a constituent of $\overline{T}_E$. Then $\overline{\pi}_v$ is again a quotient of some non-unitary principal series representation $\overline{\rho}_v(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1}(x) = |x|^{1/2}$. But now the only restriction imposed on $\mu_1^2\mu_2^2$ is that μ_1^2 and μ_2^2 both be unramified.

Let $\rho_v(\nu_1,\nu_2)$ denote the usual principal series representation of $GL_2(F_v)$ with $\nu_1\nu_2^{-1}(x) = |x|$. The definition of S_v is such that $S_v(\overline{\pi}_v) = \pi_v(\mu_1^2,\mu_2^2)$ is the class one quotient of $\rho_v(\nu_1,\nu_2)$. Thus the theory of Eisenstein series for GL_2 implies that $S(\overline{\pi}) = \otimes\, S_v(\overline{\pi}_v)$ is an automorphic representation of $\overline{G}_{\mathbb{A}}$, namely a constituent of T_E. I.e., the map S is a bijection between constituents of $\overline{T}_E$ and certain constituents of T_E. More precisely, a constituent $\pi(\nu_1,\nu_2)$ of T_E will be in the image of S if and only if ν_1 and ν_2 are globally squares of some μ_1 and μ_2.

The theory above also ties in with Kubota's computation of the eigenvalues of certain real analytic Eisenstein series. Suppose F is a totally imaginary number field and $\overline{\pi} = \otimes_v\, \overline{\pi}_v$ is a constituent of $\overline{T}_E$. For each archimedean v, $\overline{\pi}_v$ will be a complementary series representation of index $1/2$. (Only in this case is

$\rho_v(\mu_1,\mu_2)$ irreducible if $\mu_1\mu_2^{-1}(x) = |x|^{1/2}$.) For v finite, $\overline{\pi}_v$ will almost always be the class 1 quotient of the principal series representation $\rho_v(|x|^{1/4}, |x|^{-1/4})$.

Now suppose $E(\overline{g},\emptyset,s)$ is the Eisenstein series whose pole at $s = 1/2$ generates $\overline{\pi}$. Then the eigenvalues of $E(\overline{g},\emptyset,s)$ with respect to Hecke operators in $H(\overline{GL}_2(F_v),\sigma_0)$ can be computed in terms of the spherical function theory of the local representation $\overline{\pi}_v$, i.e., from Theorem 5.13. The results are in agreement with Kubota's results for $E(u,s)$ (cf. Section 1 and [21], p.53).

6.2 Construction of cusp forms of half-integral weight

Fix $F = \mathbb{Q}$ and $A = A_{\mathbb{Q}}$. Let $(\tilde{G}_A)_{gen}$ denote the set of equivalence classes of irreducible unitary genuine representations of $\tilde{G}_A$ and $G_A^\wedge$ the set of equivalence classes of irreducible unitary representations of G_A. From Section 3 it follows that certain $\tilde{\pi}$ in $(G_A)^\wedge_{gen}$ correspond to classical modular forms of weight $k/2$. It is also well-known that certain π in $G_A^\wedge$ correspond to classical modular forms of weight $k-1$.

Motivated by the correspondence between forms of weight $k/2$ and $k-1$ discovered by Shimura it is natural to introduce a general correspondence between $(G_A)^\wedge_{gen}$ and $G_A^\wedge$ following the local theory developed in Sections 4 and 5. Thus we let $\{S_p\}$ denote a family of 1-1 maps between $(G_p)^\wedge_{gen}$ (irreducible unitary genuine representations of $\tilde{G}_p$) and $G_p^\wedge$ (irreducible unitary representations of G_p) with the following three properties:

(i) for each $\tilde{\pi} \in (\tilde{G}_A)^\wedge_{gen}$, $S(\tilde{\pi}) = \otimes S_p(\tilde{\pi}_p)$ determines a well-defined irreducible unitary representation of G_A;

(ii) for almost every $\tilde{\pi}$, $S(\tilde{\pi})$ is automorphic if $\tilde{\pi}$ is automorphic;

(iii) if $\tilde{\pi}_p$ is not supercuspidal, in which case $\tilde{\pi}_p$ is naturally associated to some pair of quasi-characters (μ_1, μ_2) of $\mathbb{Q}_p^\times$, $S_p(\tilde{\pi}_p)$ is simply $\pi(\mu_1^2, \mu_2^2)$, the irreducible representation of G_p associated to the pair (μ_1^2, μ_2^2).

Implicit in (i) is the fact that S_p preserves class 1 representations (Proposition 5.16) and the fact that $\tilde{\pi}$ can be factored as $\otimes \tilde{\pi}_p$ with $\tilde{\pi}_p$ class 1 for almost every p (Theorem 3.5).

The necessity of the word "almost" in (ii) results from our having defined "automorphic representation" rather rigidly. Had we not required automorphic forms to occur in L^2 we would not have to worry about S sometimes taking square-integrable cusp

forms on $\overline{G}_{\mathbb{A}}$ to slowly increasing non-cuspidal forms on $G_{\mathbb{A}}$. See Conjecture 6.2 below and the remarks directly following it.

Note that if $\overline{\pi}$ and $S(\overline{\pi})$ are both cusp forms then the "strong multiplicity one theorem" of Casselman [4] and Miyake [27] implies that the family $\{S_p\}$ is actually uniquely determined by conditions (i)-(iii).

To prove that at least one such family $\{S_p\}$ exists one has only a finite number of "bad" primes to worry about. Indeed almost every component $\overline{\pi}_p$ of $\overline{\pi}$ is non-supercuspidal and (for such $\overline{\pi}_p$) $S(\overline{\pi}_p)$ is defined as in (iii). To prove (ii) the Selberg trace form is used. We shall treat these matters elsewhere, hopefully in the near future. In the meanwhile, we use S below to explain why constituents of $\overline{T}_0^*$ should occur in r_3 or r_1.

Note first that the restriction of S to holomorphic cusp forms must agree with Shimura's correspondence. Indeed by (iii) S_∞ takes the discrete series representation $\overline{\pi}_{k/2}$ of $\overline{G}_\infty$ to the discrete series representation π_{k-1} of G_∞. Furthermore, according to Proposition 5.17, S_p preserves eigenvalues for the zonal spherical functions as soon as $\overline{\pi}_p$ and $S(\overline{\pi}_p)$ are class 1. So if $\overline{\pi}$ corresponds to a classical modular form of weight $k/2$ with eigenvalue λ_p for $T(p^2)$ (cf. Shimura [38]) the "new form" corresponding to $S(\overline{\pi})$ is of weight $k-1$ and has eigenvalue λ_p for $T(p)$.

Now recall the correspondence

$$D^3 : \pi \to \overline{\pi} .$$

This correspondence is defined on a subset of $G_{\mathbb{A}}\hat{}$ and takes values in $(G_{\mathbb{A}})\hat{}_{gen}$. For convenience, we restrict our attention again to $\overline{T}^*$. In fact we reformulate D^3 in the context of $\overline{SL}_2$ as follows.

For each place p let $r_p(q_3, \tau_p)$ denote the Weil representation of $\overline{SL}_2(\mathbb{Q}_p)$ in $L^2(\mathbb{Q}_p^3)$ attached to q_3. We assume in

advance we have fixed a character $\tau = \prod \tau_p$ of $\mathbb{Q}\backslash\mathbb{A}$ with $\tau_\infty(x) = e^{\pi i x}$. Since τ_p is trivial on O_p for almost every p it follows from the discussion of Subsection 2.6 that $r_p(q_3,\tau_p)$ is class one with respect to $SL_2(O_p)$ for almost every p.

According to the philosophy of the Weil representation expounded in Section 2 the representation

$$r_3(\tau) = \otimes\, r_p(q_3,\tau_p)$$

is a well-defined genuine representation of $\overline{SL}_2(\mathbb{A})$ whose commuting algebra is generated by the adjoint action of the orthogonal group $PGL_2(\mathbb{A})$ in $L^2(\mathbb{Q}^3_{\mathbb{A}})$. We denote this adjoint action by A. Thus one obtains a 1-1 correspondence between irreducible constituents of $r_3(\tau)$ and irreducible representations of $G_{\mathbb{A}}$ which occur in A.

By inducing to $Z^2_{\mathbb{A}}\backslash G^2_{\mathbb{A}}$ we obtain a map (no longer 1-1) between constituents π of A and constituents $\overline{\pi}$ of r_3. This is the map we call D^3. The non-trivial fact about D^3 we use below is that

$$D^3(S(\overline{\pi})) = \overline{\pi} \tag{6.1}$$

whenever the left-hand side makes sense, i.e. $S(\overline{\pi})$ occurs in A.

The analogue of (6.1) "at infinity" was established in Subsection 4.3. One can also check (6.1) directly for all unramified p. In general, one wants to show that the functions

$$\theta(\pi,\Phi)(\overline{g}) = \int_{Z^0_\infty G_F\backslash G_{\mathbb{A}}} \varphi_\pi(h) \sum_{\xi\in\mathbb{Q}^3} r_3(\overline{g})\Phi(h\cdot\xi)\,dh$$

generate a non-trivial subspace of $L^2(\overline{X})$ which realizes $\overline{\pi}$. Here $\Phi \in \mathcal{S}(F^3_{\mathbb{A}})$ and φ_π denotes some spherical function for $\pi = S(\overline{\pi})$. For π corresponding to a classical holomorphic cusp form of weight $k-1$ such a result was obtained in [41].

Conjecture 6.2. Suppose $\overline{\pi}$ is a constituent of $\overline{T}_0^*$ with $\overline{\pi}_\infty = \overline{\pi}_{k/2}$ and $k \geq 5$. Then $\overline{\pi}$ occurs in the Weil representation r_3. In other words, every classical cusp form of weight $k/2$, $k \geq 5$, arises as a theta-series attached to a quadratic form in 3 variables.

Sketch of "Proof". The hypothesis $k \geq 5$ should make it possible to conclude that $S(\overline{\pi})$ occurs in T_0. More precisely, $S(\overline{\pi})$ is trivial on Z_A and corresponds to a cusp form of weight $k-1$. (If $k = 3$, $S(\overline{\pi})$ need not be cuspidal; cf. Shimura [38], pp. 458 and 478).

Now I want to conclude $S(\overline{\pi})$ also occurs in A. For this I need a generalized Ramanujan-Petersson conjecture which asserts that any component π_p of a cusp form π on $Z_A \backslash G_A$ with π_∞ square-integrable must occur in the regular representation of G_p in $L^2(G_p)$. The truth of this assertion for almost every p is a consequence of Deligne's proof of Weil's conjecture [5]. That the "everywhere" result should also follow from Deligne's methods was suggested to me by Langlands and later verified by Deligne.

A consequence of this Ramanujan-Petersson conjecture is that each $S(\overline{\pi}_p)$ should occur in A_p. Indeed a representation of G_p/Z_p should occur in A_p if and only if it occurs in the usual regular representation of G_p. (Cf. Lipsman [50].) For $p = \infty$ this fact is already implicit in the literature. Granting it for all p implies $S(\overline{\pi})$ occurs in A, i.e. $D(S(\overline{\pi}))$ is defined. Therefore, by (6.1), $\overline{\pi} = D(S(\overline{\pi}))$ occurs in r_3. □

Note that cusp forms of weight $1/2$ can not possibly occur discretely in r_3. Indeed the infinite component of r_3 does not contain $\overline{\pi}_{1/2}$. Thus it is natural to ask how one can construct such forms, or similar forms of weight $3/2$, which don't occur in r_3.

My conjecture is that such cusp forms occur in r_1. This is

consistent with Shimura's conjecture* quoted at the bottom of p. 1 of [40]; cf. also paragraph (C) of p. 478 of [38].

Although I have been unable to prove this conjecture I can show that all "non-trivial" constituents of r_1 at least belong to $\overline{T}_0^*$. More precisely, suppose we call a constituent $\otimes r_p^*$ of r_1 "non-trivial" if $r_p^* = r_p^e(q_1,\tau)$ for all but a non-zero even number of p. Then I can prove the following:

Theorem 6.3. Each non-trivial constituents of r_1 defines a cusp form of weight $1/2$ or $3/2$.

Remark 6.4. The belief that "odd" pieces of the basic Weil representation could determine non-trivial cusp forms was first communicated to me by Roger Howe. Howe's original observation was that Dedekind's η-function must arise this way. In our setup this means η generates a representation $\otimes r_p^*$ with r_p^* equal to $r_p^o(q_1,\tau_p)$ or $r_p^e(q_1,\tau_p)$) (according as p does or does not equal 2 or 3). Howe also surmised that η^3 should correspond to the representation $\otimes r_p^*$ with r_p^* "odd" only for p equals ∞ or 2.

Recall that

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1-e^{2\pi inz})$$

is the cusp form of weight $1/2$ whose 24-th power is the discriminant $\Delta(z)$; η^3 is a cusp form of weight $3/2$ and this is consistent with the fact that $r_\infty^* = \overline{\pi}_{3/2}$.

More remarks (concerning the nature of "non-trivial" constituents).

(1) Suppose $\otimes r_p^*$ is a "trivial" constituent of r_1. In particular, suppose r_p^* is odd for an odd number of p. Then if $\Phi \in \mathcal{S}(\mathbb{A}^3)$ belongs to the space of $\otimes r_p^*$, $\Phi(-x) = -\Phi(x)$. Consequently, by Observation 2.34, the corresponding theta-series $\theta(\Phi,g^-)$

*Note added (3/19/76). Deligne and Serre have verified Shimura's Conjecture (private communication from Deligne).

is identically zero. I.e., $\otimes r_p^*$ is indeed "trivial". The condition that r_p^* be odd for at least one p was included to exclude the representation $r_e = \otimes r_p^e(q_1,\tau)$ already discussed.

(ii) The orthogonal group of q_1 is $\{x: x^2 = 1\}$. This means an automorphic form on the orthogonal group is just a collection of representations of the p-adic groups Z_2 trivial for all but an even number of p (the global representation must be trivial on the "rational subgroup" $\{\pm 1\}$ embedded diagonally). Thus "non-trivial" constituents of r_1 correspond to "automorphic" representations of $O(q_1)$. In particular, Theorem 6.3 is consistent with the general philosophy of Subsection 2.4.

We note finally that Theorem 6.3 should ultimately appear as a special example in R. Howe's general (but as yet unpublished) theory of the Weil representation. The proof of the Theorem sketched below is not particularly elegant but it's the only one I know.

Proof of Theorem 6.3. By Theorem 4.19 the infinite component of a constituent of r_1 is $\overline{\pi}_{1/2}$ or $\overline{\pi}_{3/2}$. I.e., cusp forms occuring in r_1 are automatically of weight 1/2 or 3/2. It remains only to prove that if $r^* = \otimes r_p^*$ is a non-trivial constituent of r_1 then the theta-series

$$\theta(\overline{g},\Phi) = \sum_{\xi\in\mathbb{Q}} r^*(\overline{g})\Phi(\xi)$$

belongs to $L_0^2(SL_2(\mathbb{Q})\backslash\overline{SL_2(\mathbb{A})})$ for all appropriate Φ.

Note first that we can assume $\Phi(x) = \prod \Phi_p(x)$ with Φ_p even or odd according as r_p^* is even or odd. But by hypotheses, r_p^*, hence Φ_p, is odd for at least one p. Therefore, since $r_p^*\Phi$ is also odd, $r^*(\overline{g})\Phi(0) = 0$ and the sum in (6.2) extends only over $\xi \in \mathbb{Q}^\times$.

For each $\overline{g} \in SL_2(\mathbb{A})$, write $\overline{g} = \eta a_t \overline{k}$ with $n \in N_{\mathbb{A}}$, $a_t = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$,

$t \in \mathbb{A}^x$, and $\overline{k} = \overline{SO_2(\mathbb{R})} \prod_{p<\infty} \overline{SL_2(O_2)} = K^*$. We shall prove now that there exists some $f_0 \in \mathcal{S}(\mathbb{Q}_A)$ such that

$$|r^*(\overline{g})\Phi(\xi)| \leq |t|^{1/2} \overline{f}_0(t\xi) \tag{6.3}$$

for all $\overline{g} = na_t\overline{k}$ in $\overline{SL_2(\mathbb{A})}$. Our method of proof is based on [36].

From formulas (2.42) and (2.45) it follows that

$$|r^*(\overline{g})\Phi(\xi)| = |t|^{1/2}(r^*(\overline{k})\Phi)(t\xi) \ . \tag{6.4}$$

But the map $(\overline{g}, \Phi) \to r(\overline{g})\Phi$ defines a continuous map of $\overline{SL_2(\mathbb{A})} \times \mathcal{S}(\mathbb{Q}_A)$ into $\mathcal{S}(\mathbb{Q}_A)$. Therefore, since $\overline{K}^*$ is compact, Lemma 5 of [47] implies there exists $f_0 \in \mathcal{S}(\mathbb{Q}_A)$ such that $|r^*(\overline{k})\Phi(\xi)| \leq f_0(\xi)$ for all $\xi \in \mathbb{Q}$ and $\overline{k} \in \overline{K}^*$. This, together, with (6.4), proves (6.3).

Next we observe that for <u>any</u> $f \in \mathcal{S}(\mathbb{A})$, there exists a positive integer ℓ and a constant M such that for all $t \in \mathbb{A}^x$,

$$\sum_{\xi \in \mathbb{Q}^x} |f(t\xi)| \leq M|t|^{-2\ell}$$

Since this inequality has only to do with the definition of $\mathcal{S}(\mathbb{A})$ we omit its straightforward proof. Together with (6.3) it implies

$$\sum_{\xi \in \mathbb{Q}^x} r^*(\overline{g})\Phi(\xi) \leq M|t|^{1/2 - 2\ell} \ . \tag{6.5}$$

To conclude our proof it suffices to show that $\theta(\Phi, \overline{g})$ is bounded on (the finite measure space) $SL_2(\mathbb{Q}) \backslash \overline{SL_2(\mathbb{A})}$. For this we appeal to a well known result from reduction theory, namely that $SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) N_{\mathbb{A}} A_c^- K^*$ with $A_c^- = \{\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : |t| \leq c\}$. This result implies $\overline{SL_2(\mathbb{A})} = SL_2(\mathbb{Q}) N_{\mathbb{A}} A_c^- \overline{K}^*$. Thus, for all $\overline{g} \in SL_2(\mathbb{Q}) \backslash \overline{SL_2(\mathbb{A})}$,

$$|\theta(\overline{g}, \Phi)| \leq M|t|^{1/2 - 2\ell}$$

$$\leq M' \ ,$$

as was to be shown. □

6.3. More open problems

Numerous loose ends and open problems have already been mentioned in Sections 1 through 6. In this last paragraph I want to mention two more open problems, both non-trivial.

The first concerns a "multiplicity one" theorem for $\overline{T}_0$. Can a constituent of $\overline{T}_0$ occur more than once? For T_0 the answer is "no" ([18], Proposition 11.1.1). A related question is whether two cusp forms on $\overline{G}_{\mathbb{A}}$ must be equal if they agree at all but finitely many finite places and share the same central character? For T_0 the answer is "yes". This is the "strong multiplicity one theorem" of Casselman [4] and Miyake [27]. In classical terms the question is whether a cusp form of given weight is determined by almost all its eigenvalues for the Hecke ring and whether one can associate to this form a well-defined irreducible subspace of $L_0^2(\overline{X})$.

A second problem is to develop a theory of automorphic forms for n-fold covering groups of $GL_2(\mathbb{A})$. Such groups were constructed by Kubota in [20] and by Moore in [29].

It seems clear that most of the results of Section 3 through 5 will extend to these n-fold covering groups without essential modification (the singularities at $s = 1/2$, both for Eisenstein series and induced representations, are to be replaced by singularities at $s = 1/n$). In fact the results of Kubota described in Section 1 were originally developed by him in this generality.

The fundamental problem we encounter for n-fold covering groups is that Weil's construction is no longer available. (The cohomology class determined by Weil's representation is always of order 2.) Thus some kind of analogue for this representation corresponding to homogeneous polynomials of higher degree is needed. Although initial steps for $F = \mathbb{C}$ have been taken by Kubota in [21] much work remains to be done. The case of

p-adic fields already presents real difficulties.

Note added (March 1976). We take this opportunity to correct two formulas in [10]. The last formula displayed on p.1409 should read

$$M(s,\overline{\lambda}) = \frac{L(2s,\overline{\lambda}^2)}{L(2s+1,\overline{\lambda}^2)} \otimes M(\mu_v^*, s);$$

the integral expression above it should read

$$\frac{L(2s+1,(\mu_v^*)^2)}{L(2s,(\mu_v^*)^2)} \int_{N_v} \varphi(wng)\,dn \ .$$

References

1. Artin, E., and J. Tate, Class Field Theory. W.A. Benjamin, New York, 1968.

2. Bass, H., J. Milnor and J.-P. Serre, Solution of the congruence subgroup problem for $SL_n (n \geq 3)$ and $Sp_{2n} (n \geq 2)$, Publ. Math. I.H.E.S., 33 (1967).

3. Boerner, H., Representations of Groups, North-Holland, Amsterdam, 1963.

4. Casselman, W., On some results of Atkin and Lehner, Math. Ann. 201 (1973), pp.301-314.

5. Deligne, P., La conjecture de Weil, Publ. I.H.E.S. N° 43 (1974).

6. Gelbart, S., Automorphic Forms on Adele Groups, Annals of Mathematics Study, Princeton University Press, 1974.

7. _____, Fourier analysis on matrix space, Memoirs of the A.M.S., Number 108, 1971.

8. _____, Automorphic forms on the metaplectic group, mimeographed notes, Cornell University, July 1974.

9. _____, On the construction of modular forms of half-integral weight, 1975 U.S.-Japan Seminar on Number Theory, Ann Arbor.

10. _____, and P.J. Sally, Intertwining operators and automorphic forms for the metaplectic group, P.N.A.S., U.S.A., Vol. 72 (1975).

11. Gelfand, I.M., M. Graev, and I. Pyatetskii-Shapiro, Representation Theory and Automorphic Functions, W.B. Saunders, Philadelphia, 1969.

12. Gelfand, I.M., M. Graev, and N. Vilenkin, Generalized Functions, Volume 5, Academic Press, New York, 1966.

13. Godement, R., Notes on Jacquet-Langlands theory, Institute for Advanced Study, Princeton, 1970.

14. Goldstein, L., Analytic Number Theory, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971.

15. Harish-Chandra, Harmonic analysis on reductive p-adic groups, notes by G. VanDijk, Springer Lecture Notes, Number 162, 1970.

16. Hecke, E., Mathematische Werke, Göttingen, 1959.

17. Howe, R., The Fourier transform and germs of characters (case of Gl_n over a p-adic field), Math. Ann. 208(1974), pp.305-322.

18. Jacquet, H., and R.P. Langlands, Automorphic Forms on GL(2), Springer Lecture Notes, Volume 114, 1970.

19. Kubota, T., Ein arithmetischer Satz über eine Matrizengruppe, Journ. reine u. angew. Math., 222(1966), 55-57.

20. _____, Topological covering of SL(2) over a local field, Journ. Math. Soc. Japan, 19(1967), 114-121.

21. _____, Automorphic functions and the reciprocity law in a number field, mimeographed Notes, Kyoto University, 1969.

22. _____, Some results concerning reciprocity and functional analysis, Actes, Congrès intern. Math., 1970, Tome 1, 395-399.

23. _____, A generalized Weil-type representation, Technical Report TR 73-7, University of Maryland, January, 1973.

24. Maass, H., Lectures on modular functions of one complex variable, Tata Institute, 1964.

25. Mackey, G., Borel structures in groups and their duals, Trans. Amer. Math. Soc. 85(1957), pp.134-165.

26. _____, Unitary representations of group extensions I, Acta Math. 99(1958), pp.265-311.

27. Miyake, T., On automorphic forms on GL_2 and Hecke operators, Ann. of Math. 94(1971), pp.174-189.

28. Moore, C., Extensions and low dimensional cohomology theory of locally compact groups, I and II, Trans. Amer. Math. Soc. 113 (1964), pp. 40-63, 63-86.

29. ————, Group extensions of p-adic linear groups, Publ. Math. I.H.E.S. No. 35 (1968).

30. Niwa, S., Modular forms of half-integral weight and the integral of certain functions, Nagoya J. of Math. 56 (1975).

31. O'Meara, T., Quadratic Forms, Springer-Verlag, Berlin, New York, 1971. (2nd edition).

32. Pukansky, L., The Plancherel formula for the universal covering group of $SL(2,\mathbb{R})$, Math. Ann. 156 (1964), pp. 96-143.

33. Sally, P.J., Analytic continuation of the irreducible unitary representations of the covering group of $SL(2,\mathbb{R})$, Memoirs of the A.M.S., No. 69, 1967.

34. ————, Intertwining operators and the representations of $SL_2(\mathbb{R})$, J. of Funct. Anal. 6 (1970), pp. 441-453.

35. Shalika, J., Representations of the two-by-two unimodular group over local fields, Thesis, Johns Hopkins University, 1966.

36. ————, and S. Tanaka, On an explicit construction of a certain class of automorphic forms, Amer. J. of Math. 91 (1969), pp. 1043-1076.

37. Shimizu, H., Theta series and automorphic forms on GL(2), Journ. Math. Soc. Japan, 24 (1972), pp. 638-683.

38. Shimura, G., On modular forms of half-integral weight, Ann. of Math. 97 (1973), pp. 440-481.

39. ————, On the holomorphy of certain Dirichlet series, Proc. London Math. Soc (3) 31(1975).

40. Shimura, G., On the trace formula for Hecke operators, Acta Math., 132 (1974).

41. Shintani, T., On construction of holomorphic cusp forms of half-integral weight, Nagoya J. of Math., 58 (1975).

42. Siegel, C.L., Die Funktional gleichungen eineger Dirichletscher Reihen, Math. Zeit. 63 (1956), pp. 363-373.

43. Strichartz, R., Fourier transforms and non-compact rotation groups, Indiana U. Math. J.(6) 24(1974).

44. ————, Harmonic analysis on hyperboloids, J. Functional Anal., 12 (1973), pp. 341-383.

45. Tanaka, S., On irreducible unitary representations of some special linear groups of the second order, Osaka J. Math., vol. 3, pp. 217-227.

46. Tate, J., Fourier analysis in number fields and Hecke's zeta-functions, in Algebraic Number Theory, Tompson Book Co., Washington, D. C. (1967).

47. Weil, A., Sur certaines groupes d'operateurs unitaires, Acta Math. 111 (1964), pp. 143-211.

48. ————, Sur la formule de Siegel dans la théorie des groupes classiques, Acta Math. 113 (1967).

The following papers are also relevant to the subject matter of these Notes:

49. Asai, T., The reciprocity of Dedekind sums and the factor set for the universal covering group of $SL(2,R)$, Nagoya J. of Math., 37 (1970).

50. Lipsman, R., On the unitary representation of a semi-simple Lie group given by the invariant integral on its Lie algebra, Technical Report TR 74-64, University of Maryland, October, 1974.

51. Rallis, S. and Schiffman, G., Distributions invariantes par le groupe orthogonal, preprint (1974); also, Weil representation I: Intertwining distributions, preprint (1974).

52. Ranga Rao, R., On the Weil representation of SL_2 associated to a quadratic form, in preparation.

53. Ehrenpreis, L., The use of partial differential equations for the study of group representations, in Proceedings of Symposia in Pure Mathematics, A.M.S. Volume 26(1974), Providence, R.I.

54. Gelbart, S., and Jacquet, H., On the symmetric square of an automorphic form, in preparation.

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